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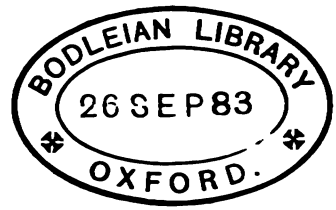

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ANDERSON, ALEX., B.A.; Queen's Coll., Galway.
ANTHONY, EDWYN, M.A.; The Elms, Hereford.
ARMENANTE, Professor; Pesaro.
BALL, ROBT. STAWELL, LL.D., F.R.S.; Professor of Astronomy in the University of Dublin.
BASU, Satish Chandra; Presidency Coll., Calcutta.
BATTAGLINI, GIUSEPPE; Professore di Matematiche nell' Università di Roma.
BELTRAMI, Professor; University of Pisa.
BERG, F. J. VAN DEN; Professor of Mathematics in the Polytechnic School, Delft.
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BICKERDIKE, C.; Allerton Bywater.
BICKMORE, C. E.; New College, Oxford.
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BOSANQUET, R. H. M., M.A.; Fellow of St. John's College, Oxford.
BOURNE, C. W., M.A.; Head Master of Bedford County School.
BROOKS, Professor E.; Millersville, Pennsylvania.
BROWN, A. CRUM, D.Sc.; Edinburgh.
BROWN, Prof. COLIN; Andersonian Univ. Glasgow.
BUCHHEIM, A., Ph.D.; Schol. New College, Oxford.
BUCK, E., B.A.; Oakhill, Bath.
BURNSIDE, W. S., M.A.; Professor of Mathematics in the University of Dublin.
CAPEL, H. N., LL.B.; Bedford Square, London.
CARMODY, W. P., B.A.; Clonmel Gram. School.
CARR, G. S., B.A.; 14 Grafton Sq., Clapham.
CASEY, JOHN, LL.D., F.R.S.; Prof. of Higher Mathematics in the Catholic Univ. of Ireland.
CAVALLIN, Prof., M.A.; University of Upsala.
CAVE, A. W., B.A.; Magdalen College, Oxford.
CAYLEY, A., F.R.S.; Sadlerian Professor of Mathematics in the University of Cambridge; Member of the Institute of France, &c.
CHAKRAVARTI, BYOMKESA, M.A.; Professor of Mathematics, Calcutta.
CHASE, PLINY EARLE, LL.D.; Professor of Philosophy in Haverford College.
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COATES, W. M., B.A.; Trinity College, Dublin.
COCHEZ Professor; Paris.
COCKLE, Sir JAMES, M.A., F.R.S.; Ealing.
COHEN, ARTHUR, M.A., Q.C., M.P.; Holland Pk.
COLSON, C. G., M.A.; University of St. Andrew's.
CONSTABLE, S.; Swinford Rectory, Mayo.
COTTERILL, J. H., M.A.; Royal School of Naval Architecture, South Kensington.
CREMONA, LUIGI; Direttore della Scuola degli Ingegneri, S. Pietro in Vincoli, Rome.
CROFTON, M. W., B.A., F.R.S.; Prof. of Math. and Mech. in the R. M. Acad., Woolwich.
CULVERWELL, E. P., B.A.; Sch. of Trin. Coll., Dubl.
CURTIS, ARTHUR HILL, LL.D., D.Sc.; Dublin.
DARBOUX, Professor; Paris.
DAVIS, J. G., M.A.; Abingdon.
DAVIS, R. F., B.A.; Wandsworth Common.
DAWSON, H. G.; Baymount, Dublin.
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DEY, Prof. NARENDRA LAL, M.A.; Calcutta.
DICK, G. R., M.A.; Fellow of Caius Coll., Camb.
DOBSON, T., B.A.; Head Master of Hexham Grammar School.
DROZ, Prof. ARNOLD, M.A.; Porrentruy, Berne.
DUPAIN, J. C.; Professeur au Lycée d'Angoulême.
EASTERBY, W., B.A.; Grammar School, St. Asaph.
EASTWOOD, G., M.A.; Saxonville, Massachusetts.
EASTON, BELLE; Lockport, New York.
EDWARDES, DAVID; Erith Villas, Erith, Kent.
ELLIOTT, E. B., M.A.; Fell. Queen's Coll., Oxon.
ELLIS, ALEXANDER J., F.R.S.; Kensington.
EMTAGE, W. T. A.; Pembroke Coll., Oxford.
ESCOTT, ALBERT, M.A.; Head Master of the Royal Hospital School, Greenwich.
ESSENNELL, EMMA; Coventry.
EVERETT, J. D., D.C.L.; Professor of Natural Philosophy in Queen's College, Belfast.
EVANS, Professor, M.A.; Lockport, New York.
FICKLIN, JOSEPH; Prof. in Univ. of Missouri.
FORTEY, H., M.A.; Bellary, Madras Presidency.
FOSTER, F. W., B.A.; Chelsea.
FOSTER, Prof. G. CAREY, F.R.S.; Univ. Coll., Lond.
FUORTES, E.; University of Naples.
GALBRAITH, Rev. J., M.A.; Fell. Trin. Coll., Dublin.
GALE, KATE K.; Worcester Park, Surrey.
GALTON, FRANCIS, M.A., F.R.G.S.; London.
GALLATLY, W., B.A.; Earl's Court, London.
GENESE, Prof., M.A.; Univ. Coll., Aberystwith.
GERRANS, H. T., B.A.; Stud. of Ch. Ch., Oxford.
GLAISHER, J. W. L., M.A., F.R.S.; Fellow of Trinity College, Cambridge.
GOLDENBERG, Professor, M.A.; Moscow.
GRAHAM, R. A., M.A.; Trinity College, Dublin.
GREENFIELD, Rev. W. J., M.A.; Dulwich College.
GREENWOOD, JAMES M.; Kirksville, Missouri.
GRIFFITH, W.; Superintendent of Public Schools, New London, Ohio, United States.
GRIFFITHS, G. J., M.A.; Fell. Ch. Coll., Camb.
GRIFFITHS, J., M.A.; Fellow of Jesus Coll., Oxon.
GROVE, W. B., B.A.; Perry Bar, Birmingham.
HADAMARD, Prof.; Paris.
HAIGH, E., B.A., B.Sc.; King's Sch., Warwick.
HALL, Professor ASAPH, M.A.; Naval Observatory, Washington.
HAMMOND, J., M.A.; Buckhurst Hill, Essex.
HARKEMA, C.; University of St. Petersburgh.
HARLEY, Harold, B.A.; King's Coll., Cambridge.
HARLEY, Rev. R., F.R.S.; Principal of Huddersfield College.
HARRIS, H. W., B.A.; Trinity College, Dublin.
HARRIS, J. R., M.A.; Clare College, Cambridge.
HART, Dr. DAVID S.; Stonington, Connecticut.
HART, H.; R.M. Academy, Woolwich.
HAUGHTON, Rev. Dr., F.R.S.; Trin. Coll., Dubl.
HAYASH, J., M.A.; Hampstead.
HENDRICKS, J. E., M.A.; Des Moines, Iowa.
HEPPEL, G., M.A.; Hammersmith.
HERBERT, A., M.A.; King Alfred's Sch., Wantage.
HERMAN, R. A., M.A.; Trin. Coll., Cambridge.
HERMITE, CH.; Membre de l'Institut, Paris.
HILL, Rev. E., M.A.; St. John's College, Camb.
HILLARY, H., M.A.; Tunbridge.
HINTON, C. H., M.A.; Cheltenham College.
HIRST, Dr. T. A., F.R.S.; Director of Studies in the Royal Naval College, Greenwich.
HOPKINS, Rev. G. H., M.A.; Stratton, Cornwall.
HOPKINSON, J., D.Sc., B.A.; Kensington.
HUDSON, C. T., LL.D.; Manilla Hall, Clifton.
HUDSON, W. H. H., M.A.; Prof. in King's Coll., Lond.
INGLEBY, C. M., M.A., LL.D.; London.
JELLY, J. O., B.A.; Magdalen College, Oxford.
JENKINS, MORGAN, M.A.; London.
JOHNSON, J. M., B.A.; Radley College, Abingdon.
JOHNSON, Prof., M.A.; Annapolis, Maryland.
JOHNSTON, SWIFT; Trin. Coll., Dublin.
JONES, L. W., B.A.; Merton College, Oxford.
KEALY, J. A., M.A.; Wilmington, Delaware.
KENNEDY, D., M.A.; Catholic Univ., Dublin.
KIRKMAN, Rev. T. P., M.A., F.R.S.; Croft Rectory, Warrington.
KITCHIN, Rev. J. L., M.A.; Heavitree, Exeter.
KITTUDGE, LIZZIE A.; Boston, United States.
KNISELY, Rev. U. J.; Newcomerstown, Ohio.
KNOWLES, R., B.A., L.C.P.; Tottenham.
LADD, CHRISTINE; Professor of Natural Sciences and Mathematics, Union Springs, New York.
LARMOR, J., M.A.; Queen's College, Galway.
LAVERTY, W. H., M.A.; Public Examiner in the University of Oxford.
LAWRENCE, E. J.; Ex-Fell. Trin. Coll., Camb.
LEIDHOLD, R., M.A.; Finsbury Park.
LEVETT, R., M.A.; King Edw. Sch., Birmingham.
LEUDESDORF, C., M.A.; Fel. Pembroke Coll., Oxon.
LOWRY, W. H., M.A.; Blackrock, Dublin.
MCADAM, D. S.; Nashington, Pennsylvania.
MCALISTER, DONALD, M.A., D.Sc.; Cambridge.
MCCAY, W. S., M.A.; Fellow and Tutor of Trinity College, Dublin.
MCCLELLAND, W. J., B.A.; Prin. of Santry School.
MCCOLL, H., B.A.; 73 Rue Stoliquin, Boulogne.
MACDONALD, W. J., M.A.; Edinburgh.
MCDOWELL, J., M.A.; Pembroke Coll., Camb.

MACFARLANE, A., D.Sc., F.R.S.E.; Examiner in Mathematics in the University of Edinburgh.
MCINTOSH, ALEX., B.A.; Bedford Row, London.
MACKENZIE, J. L., B.A.; Gymnasium, Aberdeen.
MCLEOD, J., M.A.; R.M. Academy, Woolwich.
MCVICKER, C. E., B.A.; Trinity Coll., Dublin.
MACMAHON, Capt. P. A.; R. M. Academy.
MACMURCHY, A., B.A.; Univ. Coll., Toronto.
MALET, Prof., M.A.; Queen's Coll., Cork.
MANNHEIM, M.; Professeur à l'Ecole Polytechnique, Paris.
MARKS, SARAH; Cambridge Street, Hyde Park.
MARTIN, ARTEMAS, M.A., Ph.D.; Editor & Printer of *Math. Visitor* & *Math. Mag.*, Erie, Pa.
MARTIN, Rev. H., D.D., M.A.; Edinburgh.
MATHEWS, G. B., M.A.; Leominster.
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MILLER, W. J. C., B.A., (EDITOR); The Paragon, Richmond-on-Thames.
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MONCK, HENRY STANLEY, M.A.; Prof. of Moral Philosophy in the University of Dublin.
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MORGAN, C., B.A.; Salisbury School.
MORLEY, T., L.C.P.; Bromley, Kent.
MORLEY, F., B.A.; Woodbridge, Suffolk.
MORRICE, G. G., B.A.; The Hall, Salisbury.
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O'REGAN, JOHN; New Street, Limerick.
ORCHARD, H. L., B.A., L.C.P.; Hampstead.
PANTON, A. W., M.A.; Fell. of Trin. Coll., Dublin.
PENDLEBURY, Rev. C., M.A.; London.
PHILLIPS, F. B. W.; Balliol College, Oxford.
PILLAI, C. K., M.A.; Trichy, Madras.
PIRIE, A., M.A.; University of St. Andrew's.
POLIGNAC, Prince CAMILLE DE; Paris.
POLLEXFEN, H., B.A.; Windermere College.
PRUDDEN, FRANCES E.; Lockport, New York.
PURSER, Prof. F., M.A.; Queen's College, Belfast.
PUTNAM, K. S., M.A.; Rome, New York.
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RAWSON, ROBERT; Havant, Hants.
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RILEY, R. E., B.A.; Bournemouth.
RIPPIN, CHARLES R., M.A.; Woolwich Common.
ROBERTS, R. A., M.A.; Schol. of Trin. Coll., Dublin.
ROBERTS, S., M.A., F.R.S.; Tufnell Park, London.
ROBERTS, Rev. W., M.A.; Senior Fellow of Trinity College, Dublin.
ROBERTS, W. R., M.A.; Ex-Sch. of Trin. Coll., Dub.
ROBSON, H. C., B.A.; Sidney Sussex Coll., Cam.
ROSENTHAL, L. H.; Scholar of Trin. Coll., Dublin.
ROY, Kaliprasanna, M.A.; Prof. in St. John's College, Agra.
ROYDS, J., L.C.P.; Kiveton Park, Sheffield.
RÜCKER, A. W., B.A.; Professor of Mathematics in the Yorkshire College of Science, Leeds.
RUGGERO, SIMONELLI; Università di Roma.
RUSSELL, J. W., M.A.; Merton Coll., Oxford.
RUTTER, EDWARD; Sunderland.
SALMON, Rev. G., D.D., F.R.S.; Regius Professor *of Divinity in* the University of Dublin.
SAMPSON, C. H., M.A.; Balliol College, Oxford.
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SCOTT, CHARLOTTE A.; Girton College, Camb.
SCOTT, R. F., M.A.; Fell. St. John's Coll., Camb.
SEITZ, Prof., M.A.; Kirksville, Mo., U.S.
SERRET, Professor; Paris.
SHARP, W. J. C., M.A.; Hill Street, London.
SHARPE, J. W., M.A.; The Charterhouse.
SHARPE, Rev. H. T., M.A.; Cherry Marham.
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STEWART, H., M.A.; Framlingham, Suffolk.
SWIFT, C. A., B.A.; Grammar Sch., Weybridge.
SYMONS, E. W., M.A.; Fell. St. John's Coll., Oxon.
SYLVESTER, J. J., D.C.L., F.R.S.; Professor of Mathematics in Johns Hopkins University; Member of the Institute of France, &c.
TAIT, P. G., M.A.; Professor of Natural Philosophy in the University of Edinburgh.
TANNER, Prof. H. W. L., M.A.; Bristol.
TARLETON, F. A., M.A.; Fell. Trin. Coll., Dub.
TAYLOR, Rev. C., D.D.; Master of St. John's College, Cambridge.
TAYLOR, H. M., M.A.; Fellow and Assistant Tutor of Trinity College, Cambridge.
TAYLOR, W. W., M.A.; Ripon Grammar School.
TEBAY, SEPTIMUS, B.A.; Farnworth, Bolton.
TERRY, Rev. T. R., M.A., F.R.A.S.; Fellow of Magdalen College, Oxford.
THOMAS, Rev. D., M.A.; Garsington Rect., Oxford.
THOMSON, Rev. F.D., M.A.; Ex-Fellow of St. John's Coll., Camb.; Brinkley Rectory, Newmarket.
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TODHUNTER, ISAAC, F.R.S.; Cambridge.
TOMLINSON, H.; Christ Church, Oxford.
TORELLI, GABRIEL; University of Naples.
TORRY, Rev. A. F., M.A.; St. John's Coll., Cam.
TOWNSEND, Rev. R., M.A., F.R.S.; Professor of Nat. Phil. in the University of Dublin, &c.
TRAILL, ANTHONY, M.A., M.D.; Fellow and Tutor of Trinity College, Dublin.
TROWBRIDGE, DAVID; Waterburgh, New York.
TUCKER, R., M.A.; Mathematical Master in University College School, London.
TURRELL, I. H.; Cumminsville, Ohio.
TURRIFF, GEORGE, M.A.; Aberdeen.
VINCENZO, JACOBINI; Università di Roma.
VOSE, G. B.; Professor of Mechanics and Civil Engineering, Washington, United States.
WALENN, W. H.; Mem. Phys. Society, London.
WALKER, G. F., M.A.; Queen's Coll., Camb.
WALKER, J. J., M.A.; Hampstead.
WALMSLEY, J., B.A.; Eccles, Manchester.
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WHITE, Rev. J., M.A.; Cowley College, Oxford.
WHITESIDE, G., M.A.; Eccleston, Lancashire.
WHITWORTH, Rev. W. A., M.A.; Fellow of St. John's Coll., Camb.; Hammersmith.
WICKERSHAM, D.; Clinton Co., Ohio.
WILKINS, W.; Scholar of Trin. Coll., Dublin.
WILLIAMSON, B., M.A.; Fellow and Tutor of Trinity College, Dublin.
WILSON, J. M., M.A.; Head-master, Clifton Coll.
WILSON, Rev. J., M.A.; Rect. Bannockburn Acad.
WILSON, Rev. J. R., M.A.; Royston, Cambs.
WOODCOCK, T., B.A.; Wellington, Salop.
WOLSTENHOLME, Rev. J., M.A.; Professor of Mathematics in Cooper's Hill College.
WOOLHOUSE, W. S. B., F.R.A.S., &c.; London.
WRIGHT, Dr. S. H., M.A.; Penn Yan, New York.
WRIGHT, W. E., B.A.; Herne Hill.
YOUNG, JOHN, B.A.; Academy, Londonderry.

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Solved Questions.

1100. (By the Editor.)—If a straight line be divided at random into three parts, show that the chance (1) that a triangle of any kind can be formed with the three parts is $\frac{1}{4}$; and (2) that an acute-angled triangle can be formed by the three parts is $3 \log 2 - 2$ 37

3083 & 5143. (By Dr. James Matteson.)—Find a common value of x that will make $x^2 \pm ax = \square$, $x^2 \pm bx = \square$, $x^2 \pm cx = \square$, &c., *ad inf.*; and again $a^2x^2 \pm ax = \square$, $bx^2 \pm bx = \square$, $c^2x^2 \pm cx = \square$, &c., *ad. inf.* 47

3267. (By Professor Siverly, M.A.)—Determine the axes of the maximum ellipse inscribed in the quadrant of a given ellipse 77

4387. (By Professor Cayley, F.R.S.)—Using the term "Cassinian" to denote a bi-circular quartic having four foci in a right line; show that the equation of a Cassinian having for its four foci the points $x = a$, $x = b$, $x = c$, $x = d$ on the axis of x, may be written in the four equivalent forms

$$\begin{pmatrix} . & \tau(d-c), & \sigma(b-d), & \rho(c-b) \\ \tau(c-d), & . & \rho(d-a), & \sigma(a-c) \\ \sigma(d-b), & \rho(a-d), & . & \tau(b-a) \\ \rho(b-c), & \sigma(c-a), & \tau(a-b), & . \end{pmatrix} (A^4, B^4, C^4, D^4) = 0,$$

that is, $\quad . \quad \tau(d-c)\,B^4 + \sigma(b-d)\,C^4 + \rho(c-b)\,D^4 = 0,$

$$\tau(c-d)\,A^4 \quad . \quad + \rho(d-a)\,C^4 + \sigma(a-c)\,D^4 = 0,$$

&c., &c.,

where A^4, B^4, C^4, D^4 are the distances from the four foci respectively, and the parameters ρ, σ, τ are connected by the equation

$$\rho^2(a-d)(b-c) + \sigma^2(b-d)(c-a) + \tau^2(c-d)(a-b) = 0.$$

Show also that the curve has, at right angles to the axis of x, two double tangents, the equation whereof is any one of the three equivalent forms

$$(a+d-2x)(b+c-2x) : (b+d-2x)(c+a-2x) : (c+d-2x)(a+b-2x)$$
$$= \rho^2 : \sigma^2 : \tau^2 \text{ ..... } 69$$

4417. (By S. Roberts, M.A.)—Two conics have a common focus, about which one is turned; show that the envelope of the common chords is a conic having a focus at the given focus 117

4722. (By Professor Cayley, F.R.S.)—1. Show that the conditions in regard to the reality of the roots of the equations $(x^2-a)^2 + 16A(x-m) = 0$, are, if $\quad (4m^2-3a)^3 - (8m^3 - 9ma - 27A)^2 = -,$

then the roots are two real, two imaginary; but if

$$(4m^2-3a)^3-(8m^3-9ma-27A)^2=+,$$

then, if simultaneously $a=+$, $A(ma-9A)=+$,

the roots are all real, but otherwise they are all imaginary.

2. If the roots of the foregoing equation are all imaginary, then for any real value whatever of y, the roots will be all imaginary of the equation

$$(x^2+y^2-a)^2+16A(x-m)=0 \quad \ldots\ldots\ldots\ldots\ 49$$

4667. (By the late Professor Clifford, F.R.S.)—If a conic be inverted into a circular cubic with a double point, prove that the foci and directrices of the conic will invert into foci and directing circles of the cubic 33

4862. (By Professor Martin, M.A.)—On a horizontal plane there is a hill in the form of a spherical segment, whose base is $2a$ rods in diameter, and its altitude or perpendicular height is b rods. A roadway winds round this hill, so as to be inclined at every point at angle β to the horizontal plane. Find (1) the *curve* the road is, (2) the whole length of the road, and (3) how high up the hill the road can ascend..... 44

4897. (By the late Professor Clifford, F.R.S.)—Let U, V = 0 be the point-equations, and u, $v=0$ the line-equations of the same two conics. If a tangent to U and a tangent to V are conjugate in respect of $u\pm\lambda v=0$, they will intersect on $U-\lambda^2V=0$. This last conic passes through the points of contact of the conics $u\pm\lambda v=0$ with the common tangents of u and v..... 71

4971. (By Professor Sylvester, F.R.S.)—The equation to the inverse of a conic in Cartesian coordinates being given, find the algebraical expression for the node in its simplest terms..... 34

5143. (For enunciation, see Quest. 3083) 47

5250. (By the late Dr. Booth, F.R.S.) — Through a given point within an angle draw (1) the minimum line between the sides of the given angle; and (2) show that, when the angle is a right angle, the minimum line drawn through the given point is a tangent to the quadrantal hypocycloid passing through the same point..... 85

5529. (By the late T. Cotterill, M.A.)—The curve $r=a\sin\frac{1}{\lambda}(\lambda\pi+\theta)$, where λ is an integer, being a pedal of a bi-cusped epicycloid, is unicursal; find (1) the positions of its singular points and of its four real bi-tangents; and (2) give a simple construction for tracing the curve 33

5585. (By L. H. Rosenthal, B.A.)—If we have given

$$(p^2-4r)(q^2-4t)=(pq-2s)^2,$$

(1) solve completely the sextic equation,

$$x^6+px^4+qx^3+rx^2+sx+t=0;$$

and give (2) the corresponding condition of solution, when the second term is *not* missing..... 94

5618. (By Professor Seitz, M.A.)—The perimeter of an inscriptible quadrilateral is $2a$; show that (1) its average area is $\frac{1}{8}\pi a^2\left(\frac{64}{105}-\frac{1}{8}\pi\right)$; (2) if the quadrilateral is also circumscriptible, its average area is $\frac{1}{64}\pi^2a^2$; and (3) the average area of the inscribed circle is $\frac{1}{384}\pi a^2$..... 112

5650. (By Professor Malet, F.R.S.)—Prove (1) that the locus of the poles of the sides of the triangle of reference with respect to a variable

conic which circumscribes it, and which also touches the fixed conic $ax^2+by^2+cz^2=0$, is the curve $a^{\frac{1}{2}}x^{\frac{1}{2}}+b^{\frac{1}{2}}y^{\frac{1}{2}}+c^{\frac{1}{2}}z^{\frac{1}{2}}=0$. Hence (2) show that we can inscribe in this curve a triangle whose sides pass through the vertices of the triangle of reference, any point on the curve being assumed as one vertex. Prove also (3) that, if from any point on one side of the triangle of reference we draw tangents to this curve, the line joining their points of contact passes through the opposite vertex 79

5689. (By Professor Cayley, F.R.S.)—Show (1) that the apparent contour of a Steiner's surface $(2xyz+y^2z^2+z^2x^2+x^2y^2=0)$, as seen from an exterior point on a nodal line (say the axis of z), projected on the plane of the other two nodal lines, is an ellipse passing through the four points $(\pm1, 0)$ and $(0, \pm1)$; and (2) find the surface-contour, or curve of contact of the cone and surface 31

5712. (By Professor Malet, F.R.S.) — Express in terms of Elliptic Functions the complete integral of the linear equation

$$\frac{d^3y}{dx^3}+\frac{3(1-2x)}{x(1-x)}\frac{d^2y}{dx^2}+\frac{1-7x+7x^2}{x^2(1-x)^2}\frac{dy}{dx}-\frac{1-2x}{2x^2(1-x)^2}y=0 \quad \ldots\ldots \quad 80$$

5820. (By Professor Sylvester, F.R.S.)—If $x^3+y^3+z^3=3\mu xyz$, and $w^6+y^6+z^6-x^3z^3-z^3y^3-y^3x^3=0$; show that (1) the variables will bear 18 distinct ratios to each other, if their order be taken into account; and (2) if their order be disregarded, these are reducible to the three following ratios, where we have :—

$$q=\left(\frac{27}{\mu^3}-1\right)^{\frac{1}{2}} \quad \text{and} \quad \rho^3=1.$$

$$(1+q)^{\frac{1}{2}} : (1+q\rho)^{\frac{1}{2}} : (1+q\rho^2)^{\frac{1}{2}}, \quad (1+q)^{\frac{1}{2}} : (1+q\rho)^{\frac{1}{2}}\rho : (1+q\rho^2)^{\frac{1}{2}}\rho^2,$$

$$(1+q)^{\frac{1}{2}} : (1+q\rho)^{\frac{1}{2}}\rho^2 : (1+q\rho^2)^{\frac{1}{2}}\rho,$$

[This Question contains virtually the complete analytical solution of the problem of finding the points in which a given cubic is osculable by other cubics in 9 consecutive points.] 109

6197. (By Professor Morel.) – Résoudre $x^{\frac{1}{2}}-x^{\frac{1}{3}}-7x^{\frac{1}{6}}+x^{\frac{1}{9}}+6=0$. 112

6221. (By Professor Wolstenholme, M.A.)—Trace the curve

$$\frac{12a}{\pi}\int_0^\infty \frac{\sin z \sin 2z \sin 3z \sin 4z}{z^4}\cos\frac{xz}{a}\,dz,$$

and prove that its area (between the curve and the axis of x) is $288a^2$. 53

6343. (By Professor Tanner, M.A.)—Solve the equations

$$2u_nv_n=2u_{n+1}^2=(u_n+u_{n+1})v_{n+1} \ldots\ldots 111$$

6542 & 6628. (By Professor Seitz, M.A.)—A plane is passed through three random points within a sphere; show (1) that the average area of the section is $\frac{1}{16}\pi r^2$, and (2) that the average volume of the smaller of the two segments into which the sphere is divided is $\frac{467}{1536}\pi r^3$ 43

6567. (By Professor Nash, M.A.) — In Chasles's *Aperçu Historique*, p. 342, there is given the following method of generating a Cartesian oval, or "ligne aplanétique" :—

"Etant donnés un cercle et un point fixe, pris arbitrairement dans son plan ; si par ce point on mène un rayon vecteur à un point quelconque de la circonférence du cercle, et une seconde droite, qui fasse avec un certain axe fixe un angle double de celui que fait le rayon vecteur avec cet axe,

et qu'on porte sur cette seconde droite, à partir du point fixe, un segment proportionnel au carré du rayon vecteur ; l'extrémité de ce segment aura, pour lieu géométrique, une ligne aplanétique formée de deux ovales conjuguées, dont un foyer est au point fixe."

Prove that this method of generation gives only Limaçons, that is, Cartesians in which two single foci coalesce. The fixed point is the single focus of the Limaçon, and the line joining the fixed point to the centre of the circle bisects the angle between the given line and the axis of the Limaçon. The node will be a crunode if the given point lie within the given circle, and an acnode if it lie without the circle. If the point lie in the circumference of the circle, the node becomes a cusp, and the curve a cardioid 82

6600. (By Professor Seitz, M.A.) — A plane triangle is formed by joining three points taken at random in a given semi-circumference; show that the average area (1) of the triangle is $\frac{3r^2}{\pi}\left(1-\frac{8}{\pi^2}\right)$, and (2) of its inscribed circle is $\pi r^2\left(\frac{5}{2}-\frac{162}{\pi^2}+\frac{432}{\pi^3}\right)$ 46

6691. (By Professor Hudson, M.A.) — Prove (1) that, if the curve $r = a + b\cos\theta$ have a point of inflexion, a must lie between b and $2b$; and (2) trace the change in form, as a changes from 0 to $2b$ 56

6648. (For enunciation, see Quest. 6542) 43

6731. (By W. J. C. Sharp, M.A.)—If any number of particles attract each other with a force varying directly as their masses, and inversely as the cube of the distances between them; prove that, whatever the origin and axes may be $\Sigma m(x^2+y^2+z^2) = at^2+bt+c$ 68

6756. (By E. B. Elliott, M.A.)—If u be a homogeneous function of the n^{th} degree in x_1, x_2, x_3, x_4, four variables u_1, u_2, u_3, u_4 its first, and $u_{11}, u_{12}, \ldots u_{44}$ its second partial differential coefficients, prove (1) that

$$\begin{vmatrix} u_{11}, \ldots, u_{14} \\ \ldots \quad \ldots \quad \ldots \\ u_{41}, \ldots, u_{44} \end{vmatrix} = \frac{(\quad)}{x_4^2}\begin{vmatrix} u_{11}, u_{12}, u_{13}, n_1 \\ u_{21}, u_{22}, u_{23}, u_2 \\ u_{31}, u_{32}, u_{33}, u_4 \\ u_1, u_2, u_3, 0 \end{vmatrix} + \frac{n(n-1)}{x_4^2}u\begin{vmatrix} u_{11}, u_{12}, u_{13} \\ u_{21}, u_{22}, u_{23} \\ u_{31}, u_{32}, u_{33} \end{vmatrix};$$

and (2) corresponding relations hold for any number of variables 41

6764. (By the Rev. T. R. Terry, M.A., F.R.A.S.)—Show that, unless m is a negative integer, the limit, when p is indefinitely increased, of

$$\frac{m(m+1)(m+2)\ldots\ldots(m+2p-1)}{1.3.5\ldots(2p-1)\,2m(2m+2)\ldots(2m+2p-2)} \text{ is } 2^{m-1} \ldots\ldots 42$$

6768. (By Professor Crofton, F.R.S.)—Prove that

$$f(x+hD).1 \equiv f(x) + \frac{h}{2}f''(x) + \frac{h^2}{2.4}f''''(x) + \&c. \ldots\ldots 22$$

6790. (By G. F. Walker, M.A.)—Three particles are at the corners of an equilateral triangle of side a, and are each projected parallel to the opposite side towards the same parts with a velocity $[af(a)]^{\frac{1}{2}}$, where $f(r)$ is the law of attraction between the particles; show that they will always be at the corners of an equilateral triangle 40

6809. (By J. L. McKenzie, B.A.)—(1) Let O be the orthocentre of a triangle ABC, and N the centre of the nine-point circle. Let the circle through O, and through the middle points of OB and OC, cut the line NO in P; and let the two other analogous circles cut NO in Q and R. Prove that the lines joining P, Q, and R with the middle points of BC, BA, and AB, all intersect in a point S on the circumference of the nine-points circle. (2) Let the circle through N, and through the middle points of OB and OC, cut NO in P′, and similarly for two other circles; prove that the lines joining P′, Q′, R′ to the feet of the perpendiculars on BC, CA, and AB from the opposite angles, all intersect in the same point S as before 39

6817. (By T. C. Simmons, M.A.)—From Question 6720 deduce, by reciprocation, a property of the parabola, and give an independent proof of the theorem thus obtained 64

6856. (By Professor Townsend, M.A.)—In the irrotational strain of an incompressible lamina in the plane of its mass, if the equations $r^n \cos n\theta = a^n$ and $r^n \sin n\theta = b^n$, where a and b are parameters, represent the line-systems of greatest and least displacement of the molecules; show that the equations $r^{\frac{1}{2}n} \cos \frac{1}{2}n\theta = a^{\frac{1}{2}n}$ and $r^{\frac{1}{2}n} \sin \frac{1}{2}n\theta = b^{\frac{1}{2}n}$, where a and b are parameters, represent the line-systems of greatest and least dilatation of the strain 50

6864. (By Professor Hudson, M.A.)—If I, O be the centres of the inscribed and circumscribed circles of a spherical triangle r, R their radii, and $2s = a + b + c$, prove that $2 \sin s \cos IO = (\sin a + \sin b + \sin c) \cos r \cos R$ 58

6865. (By G. F. Walker, M.A.)—Two equal particles are connected by a string which passes over a smooth peg; prove that, if one move vertically, and θ be measured from the vertical downwards, and u be the reciprocal of the radius vector, the equation of motion of the other is

$$\frac{\sin\theta}{u^3} + \frac{d}{d\theta}\left\{ \frac{\sin^2 \frac{1}{2}\theta + \frac{\sin\theta}{u}\frac{du}{d\theta}}{u^2\left(2\frac{d^2u}{d\theta^2} + u\right)} \right\} = 0 \quad \text{..... 65}$$

6872. (By R. Knowles, B.A., L.C.P.)—In a central conic, if GK is the perpendicular from G, the foot of the normal at P on CP drawn from the centre C, prove that the rectangle CP . PK is constant 38

6876. (By the Rev. W. A. Whitworth, M.A.)—Without altering the perimeter, it is required to lengthen very slightly one of the sides of an equilateral triangle and to shorten another, so that the radius of the inscribed circle may become commensurable with the sides. The increment and decrement, in order to be inappreciable to the eye, must not exceed $\frac{1}{1000}$th part of the length of the sides. Determine the ratio in which the sides may be altered 43

6878. (By T. C. Simmons, M.A.)—If A, B are two points in the plane of a given circle; give a geometrical construction for finding the position of a point P on the circumference, such that the lines AP, BP make equal angles with the tangent at P 58

6879. (By J. O'Regan.)—The horizontal base AB of a triangle being given in magnitude and position, and the side AC in magnitude only,

construct the triangle so that a body may fall along CB from C to B in the least time possible 63

6883. (By T. Morley, L.C.P.)—If the water-section of a ship be taken for the base of a right cylinder equal to the volume of the ship, and H be the height of the cylinder; prove that the vertical oscillations of the vessel are the same as the vibrations of a simple pendulum whose length is SH; S being the specific gravity of ship and cargo 68

6894. (By Professor Seitz, M.A.)—Three points are taken at random within a circle; prove that the mean value of the radius of the inscribed circle of the triangle formed by joining them is $\frac{192r}{35\pi^2}\left(\frac{128}{135}-\frac{\pi^2}{12}\right)$ ... 66

6895. (By Professor Matz, M.A.)—Find the value of ϕ in the equation
$$\pi\tan^2\phi+\sqrt{(2)}-\sqrt{(3)}=-3\left[\pi\sqrt{(6)}\tan^2\phi\right]^{\frac{1}{2}}\ldots\ldots 40$$

6917. (By J. L. McKenzie, B.A.)—A straight line XY cuts the tangents at two given points A and B on a given circle in L and M respectively, and the circle itself in C and D; then (1) if the circles ABL, ABM cut XY in α and β respectively, prove $C\alpha = D\beta$; and (2) if the circles passing through L and M, and touching the given circle in B and A respectively, cut XY in α' and β', prove $C\alpha' = D\beta'$..... 61

6924. (By Professor Seitz, M.A.)—A triangle is formed by joining three random points within a sphere; prove that the average area (1) of the triangle is $\frac{9}{77}\pi r^2$, and (2) of its inscribed circle is $(\frac{2}{15}\frac{5}{8}\frac{6}{5}\frac{1}{0}-\frac{4}{5}\frac{8}{5}\log 2)\,\pi r^2$... 60

6932. (By the Rev. T. P. Kirkman, M.A., F.R.S.)—It is an old theorem that every $(12+r)$-edron P, whose summits are all triedral, and whose faces are all 5-gons and 6-gons, has twelve pentagons. Required the number of these solids (P) for values of r up to $r=7$, no one being counted which can be drawn in any position either as the repetition or the reflected image of another; together with a complete account of symmetry, enumerating the symmetrical faces, summits, and edges. The required number is less than 20 35

6933. (By J. J. Walker, M.A.) — A homogeneous rectangular parallelepiped, the edges of which are a, b, c, floats in a liquid (the ratio of the densities of the two being $\rho : 1$) with the edge a vertical; show that, if displaced through an angle θ (the top remaining above the surface of the liquid), so that the plane of ab remains parallel to itself, the body will right itself if a^2+b^2 is less than $(1+\sec^2\theta)\div 12\rho(1-\rho)$; and hence find the condition of stability for small displacements 54

6944. (By W. Gallatly, B.A.)—Prove that, in any triangle ABC,
$$\frac{\sin\frac{1}{2}(B-C)}{\sin\frac{1}{2}(B+C)}+\frac{\sin\frac{1}{2}(C-A)}{\sin\frac{1}{2}(C+A)}+\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}$$
$$+\frac{\sin\frac{1}{2}(B-C)\cdot\sin\frac{1}{2}(C-A)\cdot\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(B+C)\cdot\sin\frac{1}{2}(C+A)\cdot\sin\frac{1}{2}(A+B)}=0\ldots\ldots 41$$

6947. (By G. F. Merritt, M.A.) — The tangent and normal to an ellipse at the point P cut the axes in T and t, G and g, respectively; GT, gt are bisected in O and o; Oo cuts the normal Pg in Q: show that PQ is one-half of the radius of curvature at P..... 120

6956. (By Professor Malet, M.A.)—If A, B, C, H, &c. be the minors of the discriminant of the quadric of which the quadriplanar equation is
$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2+2h\alpha\beta+2g\alpha\gamma+2f\beta\gamma+2l\alpha\delta+2m\beta\delta+2n\gamma\delta=0,$$

each taken with respect to the corresponding small letter, prove that the equations (1) of the quadric, (2) of the tangent line through $(\alpha', \beta', \gamma', \delta')$, (3) of the pair of tangent planes through the line joining the points $(\alpha', \beta', \gamma', \delta')$ and $(\alpha'', \beta'', \gamma'', \delta'')$, are given by equating to zero the respective determinants

$$\begin{vmatrix} A & H & G & L & \alpha \\ H & B & F & M & \beta \\ G & F & C & N & \gamma \\ L & M & N & D & \delta \\ \alpha & \beta & \gamma & \delta & 0 \end{vmatrix}, \qquad \begin{vmatrix} A & H & G & L & \alpha' & \alpha \\ H & B & F & M & \beta' & \beta \\ G & F & C & N & \gamma' & \gamma \\ L & M & N & D & \delta' & \delta \\ \alpha' & \beta' & \gamma' & \delta' & 0 & 0 \\ \alpha & \beta & \gamma & \delta & 0 & 0 \end{vmatrix},$$

$$\begin{vmatrix} A & H & G & L & \alpha' & \alpha'' & \alpha \\ H & B & F & M & \beta' & \beta'' & \beta \\ G & F & C & N & \gamma' & \gamma'' & \gamma \\ L & M & N & D & \delta' & \delta'' & \delta \\ \alpha' & \beta' & \gamma' & \delta' & 0 & 0 & 0 \\ \alpha'' & \beta'' & \gamma'' & \delta'' & 0 & 0 & 0 \\ \alpha & \beta & \gamma & \delta & 0 & 0 & 0 \end{vmatrix} \quad \text{..... 103}$$

6964. (By the Rev. T. P. Kirkman, M.A., F.R.S.)—

When the Sultan of Borneo holds a review
Upon New Year's Day, the best of the splendour
Resides in a smoking cap, jewelled and new,
Which by olden law the Vizier must render.

The gift is a fez having twenty-four faces,
Counting the pentagon girding the head royal,
Silks in a rainbow; and gorgeous laces,
With diamonds ablaze on the lines polyedrial.

Forty-two provinces quake at his name;
Forty-two scutcheons blazon their fame.
Of shields pentagonal glitter a score,
Of heptagonal six, octagonal four,
And of heralding hexagons one dozen more.

In the twenty-four shields may be any of these,
With blazoning braudric as you please,
    Aslant or askew,
    If beaded true
They star all the angles in forty-four threes.

Say now, what is royalty more than a joke,
If it cannot be lord of the fashion in smoke?

All smoke, to be loyal and orthodox,
Must kiss a right rig of the knowledge-box,
    With braiding after the King's:
(With head-gear amiss, and those flaws in your frocks,
    Your smoke and your censers
    Are bungled pretence, sirs)
Who copies a cap not a three-year-old—swings;
The old, on big dummies, fill two palace wings.

Poor Vizier! he finds it a burden sore,
To con those numberless dummy-caps o'er,
    In mortal fear and dread;
    For thus the statute of caps is read:—
If any two, placed on their pentagon brimming,
And matched apart from the pranking and trimming
Have shields in angles and order of limning
Alike o'er the base, off goes his head!

The last review is a horrible story :—
As side by side on a golden stand
Were mirror and cap, and the king was abou
To unloose the storm of his thundering band
By donning that dazzle of glory—
Treason! they shout,
And the foe rushes out,
With a battered old fez in his hand.

As slow in the mirror its image turns,
Lo! lo! into fierce and fiercer flame
In the Sultan's sight the evidence burns,
That the braided shields in order and name
On the new cap and image are all the same.

Up rose the Sultan with murderous eye,—
(I saw the Vizier begin to die),—
"O insult and wrong for a Monarch to bear!
Was ever a monster of treason so mad? O
Grandfather of asses, who bidd'st me wear
Not even a second-hand hat, but its shadow!"

Then, as trumpets and drums and yells make the coast ring,
Down goes the Vizier agape in the bow-string.

What a scandalous waste both of heads and of caps!
Mere counting of dummies might end it, perhaps,
If the tyrant could somehow politely be told
The possible number of caps, new and old.

Then find it and send it him; best, in propriety,
Signed by the Secs. of the Royal Society.

..... 118

6966. (By the Rev. W. A. Whitworth, M.A.)—Show that there are 220 triangles whose sides are integers not exceeding 100, and whose areas are also integral..... 37

6967. (By T. Woodcock, B.A.) — A ray of light passes through a medium in which the refractive power at any point is a single-valued function of the distance from a fixed point; show that all the normals that can be drawn from the fixed point to the path of the ray are situated at equal angular intervals, and have one of two lengths alternately... 123

6971. (By G. S. Carr, M.A.)—If a number of voltaic cells differing both in electro-motive force and conductivity be joined "in multiple arc," show that the difference of potentials of the electrodes of the battery, when connected by a wire, is equal to the sum of the current strengths of the separate cells divided by the sum of the conductivities of all parts of the circuit 110

6976. (By J. Young, B.A.)—Draw a common tangent to a parabola and any circle intersecting it and passing through the focus of the parabola 45

6979. (By Professor Townsend, F.R.S.)—A spiral intersecting at any constant angle all the meridians of any convex surface of revolution, being supposed the frictionless catenary of a uniform cord, or the frictionless trajectory of a material particle, constrained to rest or move on the surface, under the action of a force passing perpendicularly in every position through the axis of revolution; show that the force varies, for the catenary inversely as the square, and for the trajectory inversely as the cube, of the distance from the axis 31

6989. (By R. Tucker, M.A.)—Show that, in the curve whose equation

is $b^2e^{\frac{y}{b}} + a^2e^{-\frac{y}{b}} = 2a^2\cos\frac{x}{b}$, there are pairs of radii of curvature (ρ, ρ'), so related that $\rho^2 + \rho'^2 = a^2 + b^2$ 40

6993. (By G. Heppel, M.A.)—A conic passes through A, B, C. The tangents at those points intersect in O, P, Q. They intersect the opposite chords in T, U, V, which are in a straight line. Show that

$$\pm\frac{1}{AUV} \pm \frac{1}{BVT} \pm \frac{1}{CTU} + \frac{4}{OPQ} = \frac{1}{ABC},$$

the upper sign being taken where the lines AB, AC, &c. are produced both forwards or both backwards to meet the line TUV, the lower sign where one is produced forwards and the other backwards 123

6994. (By W. Nicholls, B.A.)—A rigid uniform beam, resting on a rough horizontal plane and against an equally rough vertical wall, is prevented from slipping by a string attached to the beam and to the intersection of the planes; the beam lying in a vertical plane perpendicular to the wall, to find the tension and the limiting position of the string ... 81

7001. (By T. Woodcock, B.A.)—The straight line through P, any point on the circumscribing circle of a triangle ABC, drawn perpendicular to the pedal line of P, meets the sides of the triangle in x, y, z: and the straight line through the orthocentre, parallel to the pedal line of P, meets the sides in X, Y, Z; Pp, Pq, Pr are drawn at right angles to PX, PY, PZ to meet the corresponding sides in p, q, r; show that (1) xp, yq, zr are bisected by the pedal line of P; (2) the triangle pqr is double of the triangle ABC; (3) PAq, PrA; PBr, PpB; and PCp, PqC are pairs of similar triangles 39

7033. (By Professor Hudson, M.A.)—A particle is attracted to one and repelled from another centre of force varying inversely as the square of the distance and of equal absolute intensity; show (1) that the surfaces of equilibrium are those formed by the revolution of parts of the curves

$$(r^2+a^2)^3(r^2+a^2-2c^2) + 4a^2r^2\cos^2\theta\,[c^4 - 2(r^2+a^2)(r^2+a^2-c^2)],$$
$$+ 16a^4r^4\cos^4\theta = 0,$$

where $2a$ is the distance between the centres of force; and find (2) what the other parts of these curves represent. 32

7034. (By the Editor.) — If ACB be a circle on AB as diameter, and P, P' two points on the vector AC such that CP = CP' = distance (CD) of C from AB; show that (1) the locus of P is a heart-shaped curve circumscribing the circle, and of P' a continuous loop inscribed therein; (2) the areas of these loci, in parts of the area of the circle, are $\frac{5}{4} \pm \frac{8}{3\pi}$; (3) the maximum vector AP_1 ($= \frac{1}{4}\sqrt{3}\,.\,AB$) makes an angle $P_1AB = 30°$ with AB; (4) the maximum tangent P_2T from P to the circle makes an angle $P_2AB = 39°\,49'\,15''$ with AB; and (5) the revolution of AP_2 round AB generates the cone of greatest entire surface that can be cut from the sphere on AB as diameter..... 86

7037. (By the Editor.) — A particle moves on the surface of an ellipsoid under the action of an attractive force varying inversely as the cube of the distance from the plane containing the greatest and least axis; show that, by a proper arrangement of the initial cir-

cumstances, the path will be one of the circular sections of ellipsoid passing through the centre 69

7047. (By R. Russell, B.A.)—Show (1) generally how to solve

$$(a_1b_1c_1f_1g_1h_1)(x, y, z)^2 = l_1, \quad (a_2b_2c_2f_2g_2h_2)(x, y, z)^2 = l_2,$$
$$(a_3b_3c_3f_3g_3h_3)(x, y, z)^2 = l_3;$$

and (2) prove that the solution of

$$x^2+yz = a, \quad y^2+zx = b, \quad z^2+xy = c,$$

depends on the solution of $u^3+v^3+w^3 = 5uvw$, where

$$u = (b-c)\left(1+\frac{\lambda}{a}\right), \quad v = (c-a)\left(1+\frac{\lambda}{b}\right), \quad w = (a-b)\left(1+\frac{\lambda}{c}\right).$$

[This is connected with the Editor's Questions 6928 and 6972.] ... 25

7060. (By R. Tucker, M.A.) — Three parabolas are described about the triangle ABC, so that A is the vertex of the diameter bisecting BC, and so on; show that (1) if p, q, r are the respective parameters of the diameters, then $\frac{1}{4}[p^{-2}+q^{-2}+r^{-2}] = 2\Sigma a^2 . \Sigma a^{-4} - 3\Sigma a^{-2}$.....(α),

$$a^4p^{-2}+\ldots = 12(a^2+b^2+c^2), \quad a^8p^{-4}+\ldots = 144(a^4+b^4+c^4)\ldots\ldots(\beta, \gamma);$$

(2) when ABC is an equilateral triangle, the area of the triangle formed by joining the foci $= \frac{4}{3}\frac{9}{4}\Delta ABC$ 53

7066. (By Dr. Macfarlane, F.R.S.E.)—Show that the number of ways in which an elector may vote, under the cumulative system of voting, when there are p candidates for q seats, is $\frac{(p+q)!}{p!\,q!}$. [Applied to the recent election of a School Board for Dundee, this result shows that the number of possible ways of voting was 43 times the population of the globe.] ... 25

7071. (By T. C. Simmons, M.A.)—Prove the two following constructions for finding the vertical chord of curvature at any point P of a parabola; (1) on PA produced take AR equal to three times the projection of the abscissa of P on AP; (2) from the focal distance SP cut off SV = AS, and on PS produced take SW = 3PS: draw a circle through the points A, V, W, meeting PA produced in R. In each of these cases, PR will be the required chord of curvature 67

7074. (By Professor Mannheim.)—On a n points fixes f, f', f''..., et le courbe (m), lieu des points tels que m, pour lequel on a $mf \times mf' \times mf''\ldots$ ... = const. Soient μ le centre de courbure de (m) pour le point m; p le point de rencontre de la normale en m à (m) et de la perpendiculaire élevée de f à fm; p' p''... des points analogues à p pour f, f', f''...

Démontrer que $\frac{1}{m\mu}\Sigma\left(\frac{1}{mp}\right) = 2\Sigma\left(\frac{1}{mp^2}\right) - \Sigma\left(\frac{1}{mf^2}\right)$..... 21

7075. (By Professor Wolstenholme, M.A.)—If in a triangle ABC there be inscribed another triangle abc, and two conics through Bca, Cab intersect again in A', B', C'; prove that the conic inscribed in ABC and touching two sides (A'B', A'C') of the triangle A'B'C' will have double contact with the conic through the five points a, b, c, B', C' 34

7081. (By the Rev. T. P. Kirkman, M.A., F.R.S.)—

"Apollo tun'd the lyre; the Muses round
With voice alternate aid the silver sound,"
Yes—that sufficed of yore; but ah!
Since roaming demigods have seen the ballet
And heard amazed the quiring storms of Halle,
Apollo swears, and Pallas vows,
That heaven must work and try to rouse
Their ever drowsier old Papa
From sleeping like a log.
So all their godships are agog
On dancing and sol-fa.

The Muses and the Charites,
By Phœbus lessoned long,
In all the lore of dance and song
Of course completely are at ease.
Not so the Heroes: many a moon
Of practising from dawn till noon
Must wax and wane, before they shine
For all the teaching of the Nine,
In either grace or tune.

The rage in heaven, of late begun,
For ballet and concert all in one,
May grow, but, as yet,
It is all quartet,
Or rather a double duet.

When Jove in his cups begins to doze,
To one couple dancing, one couple sings,
While Momus and Orpheus rouse the strings,
Ten minutes or so; and 'tis o'er
With a chorus of all at the close,
As on come another four:
And so on for hours and hours it goes.

Nor dance nor sing together again,
That night, can any coupled twain.

The Graces and Muses are all in the bill,
With gods or demigods fewer or more,
Of passable shape and skill,
Selected for training at Erato's will,
From two to many a score.

Five times each goddess divinely will sing,
Five times more enchantingly dance;
Six times each demigod's voice shall ring,
Six times that hero must prance;
All each time o'er in a different four.

The gods in a four may be all, none, or any;
(On quartets of heroes,
O spare me your *quæro's!*)
You may band any number with goddesses twelve:
But at the beginning, 'tis Pallas's pleasure,
Contriving what failure she may to shelve,
(The order in fact is
Conducive to practice,
And pushing their godships in tune and in measure,)
That two are too many,
But one may be there,
To sing to each dance of a goddess-pair.

This rule makes the riddle: this keeping in view,
Write out the programmes, say a dozen, enow
For two to twelve Heroes; then nothing is easier,
After its kind, than the task of the Vizier 118

7083. (By Dr. Macfarlane, F.R.S.E.)—Prove that

$$\frac{1}{[a^2+b^2+2ab\cos(\alpha-\beta)]^{\frac{1}{n}}}\cos n\tan^{-1}\frac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}=\frac{1}{a^n}\left\{\cos n\alpha\right.$$
$$\left.-n\frac{b}{a}\cos[(n+1)\alpha-\beta]+\frac{n(n+1)}{2!}\frac{b^2}{a^2}\cos[(n+2)\alpha-2\beta]-\&c.\right\}$$
$$=\frac{1}{(a\cos\alpha+b\cos\beta)^n}\left\{1-\frac{n(n+1)}{2!}\left(\frac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}\right)^2\right.$$
$$\left.+\frac{n(n+1)(n+2)(n+3)}{4!}\left(\frac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}\right)^4-\&c.\right\}. \quad 78$$

7084. (By J. Griffiths, M.A.)—Find the locus of intersection of three mutually perpendicular planes to an elliptic paraboloid 26

7087. (By G. S. Carr, M.A.)—If a number of voltaic elements, all differing in electro-motive force and resistance, be arranged in unequal groups, the cells of each group in multiple arc and the groups in a series; show that the current which passes through a wire connecting the poles of the battery will be given by the following rule. Multiply the sum of the separate current strengths of the cells in each group by the effective resistance of the group, and divide the sum of all the products so formed by the total effective resistance of the circuit 110

7091. (By W. J. C. Sharp, M.A.)—Show that (1) if the decimal equivalent to a fraction whose denominator is n have the greatest possible number of figures $(n-1)$ in the period, this is the same for all such fractions; (2) when this is the case, n is a prime number; and (3) no fraction with a denominator n can have $n-2$ figures in the period of the corresponding decimal 108

7099. (By the Editor.)—Sum (1) the series

$$\frac{\sin^3\theta}{1^4} - \frac{\sin^3 3\theta}{3^4} + \frac{\sin^3 5\theta}{5} - \&c.$$

between the limits $\theta = 0$, $\theta = 2\pi$; and $\theta = \frac{1}{6}\pi$, $\theta = \frac{5}{6}\pi$; and hence show (2) that $\pi^4 = \frac{3.2^4}{11}\left(\frac{1}{1^4} - \frac{1}{3^4} + \frac{1}{5^4} - \frac{1}{7^4} + \&c.\right)$ 23

7101. (By Professor Townsend, F.R.S.)—A cubic surface being supposed to have a nodal point; show that, of the three triple tangent planes intersecting in any one of its twenty-seven superficial lines, the three points of contact not on the line connect with the nodal point by a triad of conjugate diameters of the nodal cone 23

7112. (By R. F. Scott, M.A.)—Three circles, which pass through the same point O, intersect again in A, B, C. Through the point A a line is drawn meeting the circles which intersect at A in D and E. The straight lines DB, EC meet the third circle again in F and G. Prove that, as the line DAE varies, the straight line FG envelopes a circle concentric with the circle BOC 29

7116. (By A. McMurchy, B.A.)—If a_1, a_2, a_3 ... be the lengths of the sides of a convex polygon ABCD ... inscribed in a circle; p_1, p_2, p_3 ... the lengths of the perpendiculars on the considered position from any point P on the smaller arc cut off by a_1; prove that

$$\frac{a_1}{p_1} = \frac{a_2}{p_2} + \frac{a_3}{p_3} + \ldots + \frac{a_n}{p_n}$$ 30

7120. (By the Rev. T. P. Kirkman, M.A., F.R.S.)—

With the last [*i.e.*, Quest. 7081], should the riddle give any delight,
I can offer another, if Phœbus be right.
Read—Two are too many gods—one must be there,
To sing at each dance of a goddess-pair.

Terpsichore deems this cannot be done;
Minerva maintains that it may,
And never a Muse or a Grace away,
With squads heroical more than one,
Not numbering more than they.

One more, and this one cannot puzzle you long;
Urania moves it; Apollo decrees;
(Then oh! what a shout from the demigods rose!
It frightened Orion and shook the sky,
From Scorpio to Gemini):

If all the gods in dance and song—
The rules retained
Already explained—
Shall so have improved at the season's close,
As the beautiful twelve to please,
That every four that glorious night
Shall be heroes three and a goddess bright.

..... 118

7122. (By Professor Townsend, F.R.S.)—A point, taken arbitrarily

in the plane of a tetrad of circles, every two of which intersect at right angles, and three of which are consequently real while the fourth is imaginary, being supposed to undergo quadruple inversion with respect to the four circles of the tetrad taken in any order; and the several inversions being supposed performed for the twenty-four different orders in which the circles may be taken in succession; show that

(*a*) For every separate order in which the circles may be taken, the point returns after the fourth inversion to its original position in the plane.

(*b*) For the twenty-four different orders, the point and its several inverses form a group of eight points divisible into pairs of concyclic tetrads in six different ways 116

7125. (By Professor Wolstenholme, M.A.)—Prove that, if

$$y = \frac{1}{e^x - 1}, \text{ then } \frac{d^n y}{dx^n} = (-1)^n \left\{ \frac{A_1}{e^x - 1} + \frac{A_2}{(e^x - 1)^2} + \ldots + \frac{A_{n+1}}{(e^x - 1)^{n+1}} \right\},$$

where $A_p = \Delta^p 0^n + \Delta^{p-1} 0^n$ 26

7166. (By Professor Townsend, F.R.S.) — The central axis of a system of three forces in a common space being supposed to pass through the centre of the quadric determined by their three lines of action in the space; show, on elementary principles, that it must coincide with a principle axis of the surface 85

7167. (By Professor Minchin, M.A.) — A rigid body has uniplanar motion; find the locus, at any instant, of points the directions of whose velocities make a given angle with the directions of their accelerations. 22

7171. (By E. W. Symons, M.A.)—Prove that, in any triangle,

$$\cot^2 A + \cot^2 B + \cot^2 C > 1 \text{ ..... } 29$$

7175. (By G. G. Morrice, B.A.)—Prove that the angle between any two straight lines in space $= \sin^{-1} | l_1, m_2, n_3 |$, where $l_1 m_1 n_1$, $l_2 m_2 n_2$ are their direction-cosines, and $l_3 m_3 n_3$ those of their shortest distance 28

7179. (By the Rev. G. Richardson, M.A.)—If AOD be the perpendicular from A on the opposite side of the triangle ABC, O being the orthocentre, I the centre of the inscribed circle, r its radius; prove that $IO^2 = 2r^2 - AO \cdot OD$ 27, 99

7181. (By H. L. Orchard, M.A.)—Given that the sum of the radii of the inscribed, escribed, and circumscribed circles of a certain equilateral triangle is unity, find the area of the triangle 28

7185. (By Morgan Jenkins, M.A.)—Show, as a special case of Gergonne's general construction for describing a circle to touch three given circles (Salmon's *Conics*, Ch. ix., Art. 127), that the nine-points-circle of a triangle touches the three escribed circles of that triangle; also, in like manner, that it touches two of the escribed circles and the inscribed circle; show also that the three circles, each of which touches two of the escribed circles externally and one internally, pass through the same point, viz., the centre of the circle inscribed in the triangle obtained by joining the middle points of the sides of the first triangle 88

7186. (By R. Tucker, M.A.)—If A′, B′, C′, (D, E, F) are the mid-points respectively of the arcs (and sides) BC, CA, AB of a circle (and triangle) ABC; prove that (1) the pedal lines of A′, B′, C′ with reference to the triangle ABC form a triangle A″B″C″ whose sides are equal and parallel to the sides of A′B′C′, and A″D, B″E, C″F meet in the ortho-

centre of A″B″C″; and (2) if P, P′ are any two diametral points, their four pedal lines with reference to the triangles ABC, A′B′C′ form a rectangle..... 121

7188. (By Dr. Curtis.)—Prove that the centre of gravity of the area included by a loop of the curve $r = a\cos n\theta$ is at a distance from the origin

$$= 16an^4 \sin\left(\frac{n+1}{2n}\pi\right) \div \left\{(n^2-1)[(3n)^2-1]\pi\right\}.$$

[The cases for $n = 2$ and $n = 3$ are exercises in Todhunter's *Statics*.] 27

7197. (By the Editor.)—With two given sides, construct a triangle such that the square of the sine of half the angle contained by the sides may be equal to the squares of the sines of the other two semi-angles. 56

7200. (By G. S. Carr, B.A.)—If two non-intersecting circles, having radii a, b, centres A, B, and distance between the centres c, be inverted into concentric circles having their common centre at a distance d from A in the direction AB; show that α, β, the radii of the images, must have the values

$$\alpha = \frac{(2cd - a^2 + b^2 - c^2 + S)\,a}{a^2 - b^2 + c^2 - S}; \quad \beta = \pm\frac{(2cd - a^2 + b^2 - c^2 + S)\,b}{a^2 - b^2 - c^2 - S},$$

where $$S = \sqrt{(a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2)},$$

and the + or − sign of β is to be read according as the smallest circle, radius b, lies within or without the other 63

7201. (By R. F. Scott, M.A.)—Prove that

$$\int_0^{\pi} x \log(1 - \sin^2\alpha \sin^2 x)\,dx = 2\pi^2 \log\cos\tfrac{1}{2}\alpha \ldots\ldots 54$$

7203. (By W. J. C. Sharp, M.A.)—In the late Professor Clifford's Quest. 6009 (*Reprint*, Vol. xxxiii., p. 59),—as to what is the form of a kite string,—assuming that the force of the wind is a function of the height, show how this function may be determined, so that the string shall assume a given form 99

7204. (By J. Hammond, M.A.)—If

$$\phi(x) = \phi\left(\frac{1}{x\sin^2\alpha + \cos^2\alpha}\right), \quad \int_0^{\infty} \frac{\phi(x)\,dx}{x^2\tan^2\alpha + x - \sec^2\alpha} = 0,$$

$$\int_0^{\infty} \frac{\phi(x)\,dx}{(x-1)^2} = (1 + \sin^2\alpha)\int_0^1 \frac{\phi(x)\,dx}{(x-1)^2} \quad \ldots\ldots 98$$

7208. (By A. McMurchy, B.A.)—If n be the length of a normal chord to a surface $ax^2 + by^2 + cz^2 = 1$ at a point for which r is the distance from the centre, and p the perpendicular on the tangent plane, prove that

$$\frac{2}{np^3} - \frac{a+b+c}{p^2} + bc + ca + ab - abcr^2 = 0 \quad \ldots\ldots 55$$

7213. (By C. Morgan, B.A.)—Show that, in any triangle,

$$s \cdot \Sigma\left(r_a^{-1}\right) = \cot\tfrac{1}{2}A \cot\tfrac{1}{2}B \cot\tfrac{1}{2}C,$$

where (as usual) r_a is the radius of the circle escribed to the side a, and s is the semiperimeter 30

7215. (By A. H. Curtis, LL.D., D.Sc.)—Prove that

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\&c.\ldots\ ad\ inf.=\frac{\pi^4}{90},$$

$$\frac{1}{1^4}-\frac{1}{2^4}+\frac{1}{3^4}-\&c.\ldots\ ad\ inf.=\frac{(2^3-1)\,\pi^4}{2^3\times 90}=\frac{7\pi^4}{720},$$

$$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\&c.\ldots\ ad\ inf.=\frac{2^4-1}{2^4}\times\frac{\pi^4}{90}=\frac{\pi^4}{96};$$

$$\frac{1}{1^8}+\frac{1}{2^8}+\frac{1}{3^8}+\&c.\ldots\ ad\ inf.=\frac{\pi^8}{3.5.7.90}=\frac{\pi^8}{9450},$$

$$\frac{1}{1^8}-\frac{1}{2^8}+\frac{1}{3^8}-\&c.\ldots\ ad\ inf.=\frac{2^7-1}{2^7}\times\frac{\pi^8}{9450}=\frac{127}{128}\times\frac{\pi^8}{9450},$$

$$\frac{1}{1^8}+\frac{1}{3^8}+\frac{1}{5^8}+\&c.\ldots\ ad\ inf.=\frac{2^8-1}{2^8}\times\frac{\pi^8}{9450}=\frac{255}{256}\times\frac{\pi^8}{9450}=\frac{17\pi^8}{256\times 630}$$

..... 59

7217. (By T. Muir, M.A., F.R.S.E.)—Prove the identity

$$\begin{vmatrix} x & 0 & 0 & x_1^2 & x_1x_2 & x_2^2 \\ 0 & y & 0 & y_1^2 & y_1y_2 & y_2^2 \\ 0 & 0 & z & z_1^2 & z_1z_2 & z_2^2 \\ 0 & z & y & 2y_1z_1 & y_1z_2+y_2z_1 & 2y_2z_2 \\ z & 0 & x & 2z_1x_1 & z_1x_2+x_1z_2 & 2z_2x_2 \\ y & x & 0 & 2x_1y_1 & x_1y_2+x_2y_1 & 2x_2y_2 \end{vmatrix} = |\,xyz\,|^3 \quad \ldots\ldots\ 106$$

7218. (By Professor Genese, M.A.)—Prove that

$$\begin{vmatrix} x & 0 & 0 & x_1^2 & x_2^2 & l \\ 0 & y & 0 & y_1^2 & y_2^2 & m \\ 0 & 0 & z & z_1^2 & z_2^2 & n \\ 0 & z & y & 2y_1z_1 & 2y_2z_2 & 2p \\ z & 0 & x & 2z_1x_1 & 2z_2x_2 & 2q \\ y & x & 0 & 2x_1y_1 & 2x_2y_2 & 2r \end{vmatrix} = -2\Delta\{lX_1X_2+\ldots+p(Y_1Z_2+Z_1Y_2)+\ldots\}$$

where $\Delta=\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$, X_1 is the minor of x_1 in Δ, &c.

..... 106

7219. (By Professor Sylvester, F.R.S.) — Prove that a series of continually increasing integers, no one of which is a sum of positive multiples of all or any of those that precede it, cannot be continued *ad infinitum* 85

7221. (By Professor Townsend, F.R.S.)—In the irrotational strain of an incompressible substance in the space of its mass, if the equipotential surfaces of the strain be a system of confocal ellipsoids in the space, determine the form of the potential as a function of the parameter of the system 102

7224. (By Professor Cochez.)—Démontrer que la fonction

$$ke^{2x}\left\{[(x+y)^2+c]^{\frac{1}{2}}-x-y\right\}=[(x+y)^2+c]^{\frac{1}{2}}+x+y,$$

est telle que $x+y=y''$ 57

7225. (By Professor Cavallin, M.A.)—If a closed convex curve glides always in contact with two fixed straight lines, and rotates until returning to its original position; prove that the mean value of the arc com-

prised between the points of contact is the same for all curves that have the same perimeter..... 92

7228. (By Professor Hudson, M.A.)—From a point A on the outer of two concentric circles, tangents AP, AQ are drawn to the inner; AP, QP meet the outer again in T, R; prove that RP : RQ = RT² : RA². 66

7229. (By Professor Martin, M.A., Ph.D.)—Show that the value of the integral $\int_0^1 [(1+e^2)\,\mathrm{E}(e,1)-(1-e^2)\,\mathrm{F}(e,1)]\,de$, where F $(e, 1)$ and E $(e, 1)$ denote complete elliptic functions of the first and second kinds, is ·72445 93

7232. (By R. F. Scott, M.A.)—If (a, b, c) stand for the straight line $ax+by+cz=0$, show that the six lines obtained by permutating a, b, c in (a, b, c) in all possible ways, all touch the conic

$$\frac{x^2+y^2+z^2}{a^2+b^2+c^2}-\frac{2(yz+zx+xy)}{a^2+b^2+c^2-2bc-2ca-2ab}=0 \quad \ldots\ldots\ldots 75$$

7233. (By G. S. Carr, B.A.)—If two non-intersecting circles, having radii a, b, centres A, B, and distance between the centres c, be inverted into concentric circles with radii α, β, and centre O; show that, if we denote $(a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2)^{\frac{1}{2}}$ by S, we have

$$\mathrm{AO}=\frac{a^2-b^2-c^2-\mathrm{S}}{2c}+\frac{k^2c}{\mathrm{S}},\quad \alpha=\frac{2k^2c^2a}{\mathrm{S}(a^2-b^2+c^2-\mathrm{S})},\quad \beta=\frac{\pm 2k^2c^2b}{\mathrm{S}(a^2-b^2-c^2-\mathrm{S})},$$

a being $>b$ and the + or − sign being read according as the smallest circle is within or without the largest 63

7235. (By C. Leudesdorf, M.A.) — Each of three circles (radii ρ_1, ρ_2, ρ_3) touches two sides of a plane triangle ABC and the nine-point circle, having external contact with the latter; if R, r be the radii of the circumscribed and inscribed circles of ABC, and Δ its area, prove that

$$(r-\rho_1)(r-\rho_2)(r-\rho_3)=2\mathrm{R}^2r^5\Delta^{-2} \quad \ldots\ldots\ldots 96$$

7238. (By W. J. C. Sharp, M.A.)—Prove that the circumscribed circle of a triangle bisects the lines OO_1, OO_2, OO_3, $\mathrm{O}_2\mathrm{O}_3$, $\mathrm{O}_3\mathrm{O}_1$, $\mathrm{O}_1\mathrm{O}_2$, where O, O_1, O_2, O_3 are the centres of the inscribed and circumscribed circles 77

7237. (By W. J. C. Sharp, M.A.)—If cones be constructed having their vertices at two given points and enveloping a given quadric; prove that (1) their common curve will consist of two conics lying on two planes which pass through the intersection of the polar planes of the given points; and (2) if these points lie on a quadric concentric, similar, and similarly situated to the given one, one of the planes will pass through the centre, and the other will be the polar plane of the middle point of the line joining the given points 84

7239. (By G. Turriff, M.A.)—The angle B and the exterior angle A of a triangle ABC are bisected by straight lines meeting in G. With centre C and radius equal to OG, a circle is drawn cutting GA, GB, or these produced, in D, E. Prove that DE is parallel to AB 76

7241. (By A. McMurchy, B.A.)—Show that, if m, n, p, q be positive

integers, the limiting value, when $x = y = z = a$, of the fraction

$$\frac{x^m(y^n-z^n)+y^m(z^n-x^n)+z^m(x^n-y^n)}{x^p(y^q-z^q)+y^p(z^q-x^q)+z^p(x^q-y^q)}, \text{ is } \frac{mn(m-n)}{pq(p-q)}a^{m+n-p-1}\ldots \quad 76$$

7242. (By S. Tebay, M.A.)—*Ring-leaping.* The gymnast will know the meaning of this phrase. Two hand-rings are suspended from a beam by cords. The leaper sets himself in motion by swinging to and fro, and when this has become sufficiently lively he quits the rings and alights on the ground. Find the greatest distance he can leap 101

7246. (By Capt. MacMahon, R.A.)—From any point P in the side BC of the triangle ABC, lines PD, PE are drawn parallel to AC and AB, meeting those sides in D and E; prove that the envelope of DE is a parabola, whose focus is at a distance from A equal to $bc\,[2(b^2+c^2)-a^2]^{-\frac{1}{2}}$. 74

7249. (By Professor Sylvester, F.R.S.)—If

$$1-\frac{1}{1-c}y+\frac{c}{(1-c)(1-c^2)}y^2+\ldots\pm\frac{c^{\frac{1}{2}(n^2-n)}}{(1-c)(1-c^2)\ldots(1-c^n)}y^n+\ldots=0,$$

prove that (1) the sum of the i^{th} powers of the several values of $\frac{1}{y}$ is $\frac{1}{1-c^i}$; and (2) the sum of the i^{th} homogeneous products of the same values is $\frac{1}{(1-c)(1-c^2)\ldots(1-c^i)}$.

[If $i = 2$, the first sum is $\left(\frac{1}{1-c}\right)^2-2\frac{c}{(1-c)(1-c^2)}$, which is $\frac{1}{1-c^2}$, and the second sum is $\left(\frac{1}{1-c}\right)^2-\frac{1}{(1-c)(1-c^2)}$, which is $\frac{1}{(1-c)(1-c^2)}$.]
..... 50

7255. (By Professor Martin, M.A., Ph.D.)—An urn contains 12 balls, of which either 5, or 6, or 7, or 8, are white (but which one of these numbers is not known), and the rest are black; if a ball is drawn at random, find the chance that it is white 51

7257. (By Professor Malet, F.R.S.)—A cone of the second degree intersects a given sphere in two planes, one fixed, and the other touching in every position another fixed sphere concentric with the first; prove that the locus of the vertex of the cone is a pair of spheres 80

7258. (By the Editor.)—If a square board, of side a, be supported in a horizontal position by two equal loops of string, of length λ, that pass under opposite corners, and are suspended from a fixed hook; prove that the tensions of the strings are

$$\frac{W}{4}\cdot\frac{\lambda-a\sqrt{2}}{(\lambda^2-2\sqrt{2}a\lambda)^{\frac{1}{2}}}. \quad \ldots\ldots 115$$

7259. (By Professor Minchin, M.A.) — A point P describes an orbit, with acceleration f, directed always to a fixed centre O; prove that the acceleration of the corresponding point P′, of the Hodograph, is in general, non-central, and makes with the radius vector OP′ an angle χ given by the equation $\tan\chi=\dfrac{\frac{d}{d\theta}\log\frac{f}{r}}{1-\cot\phi\,\frac{d}{d\theta}\log f}$, where ϕ is the angle between OP and the tangent at P, r = OP, and θ is the polar angle of r

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7262. (By E. B. Elliott, M.A.)—Two bicircular curves of the fourth class have the foci of a given conic Σ for double foci, and have each a pair of tangents to Σ for double tangents. Prove that every common tangent of the two, except the tangent at the circular points, passes through one or other of two fixed points dividing harmonically the distance between the intersections of the two pairs of tangents to Σ 103

7263. (By J. J. Walker, M.A.) — Show that the principal axes through any point (O) of a homogeneous solid, with respect to which the bounding surface (S) is convex, are also principal axes for the shell (S) supposed to have at each point a surface density proportional to the perpendicular from (O) on the tangent plane at that point..... 62

7266. (By Dr. Macfarlane, F.R.S.E.)—Find the number of different possible states of the poll, under the cumulative system of voting, when there are p candidates, q seats, and r electors 94

7267. (By C. Morgan, B.A.)—If arcs of great circles be drawn from the angles A and B of a spherical triangle perpendicular to the opposite sides; prove that, if $\tan\alpha\tan\alpha' = \tan\beta\tan\beta'$, where $\alpha, \alpha', \beta, \beta'$ are the segments into which the sides are divided, the triangle must be either isosceles or right-angled 96

7269. (By Belle Easton.)—One of two bags contains 10 sovereigns, and the other 10 shillings. One coin is taken out of each and placed in the other. This is repeated ten times. Show that now the expectations of these bags are $105 \pm 95\,(\frac{4}{5})^n$ 71

7272. (By R. F. Scott, M.A.)—If S and H are the foci of a system of confocal conics, and on SH as diameter a circle is drawn; prove (1) that the locus of the extremities of the diameters conjugate to those drawn to the points where the circle meets the conics of the system, is a lemniscate of Bernoulli; and (2) find what the locus becomes for a circle concentric with the given circle..... 82

7277. (By Professor Sylvester, F.R.S.)—If S_k denotes the sum of the k-ary products of 1, $\frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{i-1}$, prove that the coefficient of any positive $(1+k)^{\text{th}}$ power of $\frac{1}{x}$, in the development of $\frac{1}{(1-e^{-x})^i}$ in a series of ascending powers of x, is $\Pi k \,.\, S_k$ 74

7279. (By Professor Hudson, M.A.)—On a given base AB a right-angled spherical triangle ACB is drawn; on CA, CB, CP, CQ are taken each equal to a quadrant; find the loci of P and Q 79

7280. (By Professor Malet, F.R.S.)—If any three conjugate semi-diameters of the ellipsoids

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{k^2}, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{k'^2},$$

meet the three surfaces respectively in the points P, Q, R, prove that the envelope of the plane PQR is the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{1+k^2+k'^2} \quad \ldots\ldots 73$$

7283. (By Professor Martin, M.A., Ph.D.) — From a full cask of

wine a quantity is drawn off at random, and the cask filled up with water. Find the chance that, after n such operations, the cask contains more wine than water 114

7285. (By the Editor.)—From every point in a given curve a straight line is drawn, making with the ordinate on one side the same angle that the tangent makes with the ordinate on the other side: find (1) the locus of the intersections of these lines; and (2) prove that, if the given curve is the common parabola, the locus will be a parabola whose latus-rectum is triple that of the first 72

7286. (By W. J. C. Sharp, M.A.)—Prove that perpendiculars drawn through the middle points of the sides of a triangle to the inner bisectors of the angles are the radical axes of the inscribed and the several escribed circles 79

7289. (By R. Warburton-White, B.A.)—Show that the nine-point circle touches the inscribed and escribed circles of a triangle..... 99

7299. (By R. Knowles.)—P is any point on an ellipse of which S and H are the foci; circles are drawn on SP and HP as diameters; show that (1) the loci of their centres are two equal ellipses intersecting on the minor axis of the given ellipse; (2) the circles touch the auxiliary circle; (3) the line joining the points of contact touches the ellipse at P 73

7302. (By Professor Townsend, F.R.S.)—If two orthogonal systems of lines in the plane of an incompressible lamina be the potential and displacement line-systems of a possible irrotational strain of the lamina; show that—

(a) The two inverse systems of lines with respect to any arbitrary circle, real or imaginary, in the plane will be those of another possible irrotational strain of the lamina.

(b) At all pairs of corresponding points of the two inverse strains, the ratios of the two actual displacements and of the two principal dilatations are proportional respectively to those of the simple and those of the squared inverse distances from the centre of the circle 97

7303. (By Professor Morel.)—Si A, B, C, a, b, c sont les éléments d'un triangle, et ϕ un angle satisfaisant à la relation

$$\frac{b^2+c^2-2bc\cos(A-2\phi)}{b^2+c^2-2bc\cos A} = \frac{c^2+a^2-2ca\cos(B-2\phi)}{c^2+a^2-2ca\cos C};$$

(1) l'un quelconque de ces rapports est aussi égal à

$$\frac{a^2+b^2-2ab\cos(C-2\phi)}{a^2+b^2-2ab\cos C};$$

et en outre (2) on a, en supprimant la solution $\phi = 0$, la relation

$$\cot\phi = \cot A + \cot B + \cot C \text{ ..... } 108$$

7307. (By Professor Wolstenholme, M.A.)—A given rigid body, of mass m, is kinetically equivalent to a system of four equal particles (each of mass $\frac{1}{4}m$) placed at the corners of any maximum tetrahedron inscribed in the ellipsoid $\dfrac{x^2}{B+C-A}+\dfrac{y^2}{C+A-B}+\dfrac{z^2}{A+B-C}=\dfrac{3}{2}$,

where MA, MB, MC are the moments of inertia about the principal axes at the centroid, which are also the axes of coordinates. If the notation,

$MA = \Sigma(mx^2)$, $MB = \Sigma(my^2)$, $MC = \Sigma(mz^2)$, be used, the equation of the ellipsoid will be $\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 3$ 105

7309. (By W. J. C. Sharp, M.A. Suggested by Question 7249.)—If S_r denote the value of the sum of the r^{th} homogeneous products of any quantities, and s_r the sum of the r^{th} powers of the same quantities; show that $S_1 = s_1$, $2S_2 = S_1 s_1 + s_2$, $3S_3 = S_2 s_1 + S_1 s_2 + s_3$,

and $rS_r = S_{r-1}s_1 + S_{r-2}s_2 + S_{r-3}s_3 + \ldots + s_r$ 95

7311. (By J. J. Walker, M.A.)—If a point P describes a circumference of radius a with a velocity varying as the square of the distance (r) from any fixed point in its plane, show that the acceleration makes with the circumference at P an angle ψ determined by

$$\cot\psi = -2a\cos\phi \div r,$$

ϕ being the angle which r makes with the circumference at P; and that the acceleration varies as $r^3\cos\phi \div \cos\psi$ 93

7312. (By R. Tucker, M.A.)—Along the axis of a parabola ($y^2 = 9ax$) distances $AK = KL = LM = a$ are measured from the vertex (A), and the double ordinates BKB', CLC', DMD' are drawn; on DD', lengths $MR = MS = a$ are taken, and lines RP, SP, through any point P on CC', meet BB' in T, T'. Prove that perpendiculars to ST, RT', at T, T', meet on the parabola. [This theorem is susceptible of considerable generalization by assigning different lengths to AK, KL, LM, MR, MS.] 97

7313. (By C. Leudesdorf, M.A.)—Prove that the equations

$$\frac{ad-bc}{2(ac-b^2)} = \frac{ae+2bd-3c^2}{3(ad-bc)} = \frac{3(be-cd)}{ae+2bd-3c^2} = \frac{2(ce-d^2)}{be-cd}$$

are equivalent to $ae + 3c^2 - 4bd = 0$, $ace + 2bcd - ad^2 - eb^2 - c^3 = 0$..... 91

7315. (By W. S. M'Cay, M.A.)—If XX', YY', ZZ' be the diagonals of a quadrilateral circumscribed to a circle, L, M, N their mid-points, and O the centre of the circle; prove that

$$\frac{OX \cdot OX'}{OL} = \frac{OY \cdot OY'}{OM} = \frac{OZ \cdot OZ'}{ON}$$ 88

7320. (By R. Knowles, B.A.) — If n_r be the coefficient of x^r in the expansion of $(1+x)^n$, prove that

$$u = n_1 - \frac{2^2-1}{2}n_2 + \frac{2^3-1}{3}n_3 - \ldots + (-1)^{n-1}\frac{2^n-1}{n}n_n$$

$$= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \ldots + (-1)^n\frac{1}{n}$$ 102

7322. (By Professor Sylvester, F.R.S.)—Prove that the number of sequences of consecutive numbers (a "sequence" of one term not excluded) having the sum n, is equal to the number of odd divisors of n.

Ex.—The odd divisors of 9 are 1, 3, 9, and 9 is represented by the three sequences 2, 3, 4; 4, 5; 9. The odd divisors of 15 are 1, 3, 5, 15, and 15 is represented by the four sequences 1, 2, 3, 4, 5; 4, 5, 6; 7, 8; 15.

[The extension of the theorem is as follows: n can be represented by *pairs* of sequences of consecutive numbers having no common term in as many ways as there are ways of satisfying the equation in positive integers $x(2y+1) + z(2t+1) = n$. Again, n can be represented by as many *triads*

of sequences having no common term as there are solutions or positive integers of the equation $x(2y+1)+z(2t+1)+u(2v+1)=n$, and so on.] 122

7324. (By Professor Wolstenholme, M.A.) — A triangle A'B'C', whose angles are given, circumscribes a given triangle ABC; prove that there will be two such triangles of maximum area; and, if these areas be Δ_1, Δ_2, and R the radius of the circle ABC,

$$\Delta_1 \sim \Delta_2 = 8R^2 \sin A \sin B \sin C,$$

$$\Delta_1 + \Delta_2 = 4R^2(\sin^2 A \cot A' + \sin^2 B \cot B' + \sin^2 C \cot C').$$

[If the *sign* of the areas be reckoned according to the usual rule (that A'B'C' is positive when, in going round the triangle in the order A', B', C', the area is on the left hand), the smaller of these will be negative, and with this convention $\Delta_1 + \Delta_2 = 8R^2 \sin A \sin B \sin C$,

$$\Delta_1 - \Delta_2 = 4R^2(\sin^2 A \cot A' + \ldots + \ldots).] \ldots\ldots\ldots\ldots 124$$

7327. (By Professor Wolstenholme, M.A.) — Prove that (1) the maximum value of $f(x) \equiv (1+x)^{\frac{1}{2}(1+x)}(1-x)^{\frac{1}{2}(1-x)} + (1-x^2)^{\frac{1}{2}} - 2$ lies between ·093511 and ·093512; and (2) the corresponding value of x is ·9653863 very nearly..... 109

7328. (By the Editor.)—Show that the locus of the vertex of a triangle on a given base, when the centre of the inscribed square moves on a given conic, is another conic 117

7330. (By the Editor.) — Prove that the area of a triangle whose sides are roots of the equation $f(x) \equiv x^3 + px^2 + qx + r = 0$, is

$$\Delta = \tfrac{1}{8} p^{\frac{1}{2}} (4pq - p^3 - 8r)^{\frac{1}{2}} \ldots\ldots\ldots\ldots 111$$

7338. (By S. Tebay, B.A.)—The parallel sides of a trapezoid are $m^2 + 2mn - n^2$ and $n^2 + 2mn - m^2$; show that it is divided into two equal trapezoids by the line $m^2 + n^2$ 86

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

7074. (By Professor MANNHEIM.)—On a n points fixes f, f', f'' ..., et le courbe (m), lieu des points tels que m, pour lequel on a $mf \times mf' \times mf'' \ldots$ $\ldots$ = const. Soient μ le centre de courbure de (m) pour le point m; p le point de rencontre de la normale en m à (m) et de la perpendiculaire élevée de f à fm; p', p'' ... des points analogues à p pour f, f', f'' ...

Démontrer que $$\frac{1}{m\mu}\,\Sigma\left(\frac{1}{mp}\right) = 2\Sigma\left(\frac{1}{mp^2}\right) - \Sigma\left(\frac{1}{mf^2}\right).$$

Solution by W. J. C. SHARP, M.A.; CHRISTINE LADD, B.A.; *and others.*

Denote the distances mf, mf', &c., by r_1, r_2, &c., the angles which these make with a prime radius by θ_1, θ_2, &c., and those they make with the tangent by ψ_1, ψ_2, &c.; then $r_1 \ldots r_n = c$; hence, differentiating,

$$\Sigma\frac{dr_1}{r_1} = 0, \quad \text{or} \quad \Sigma\frac{\cos\psi_1}{r_1} = 0,$$

since $dr_q = \cos\psi_q\, ds$. Differentiating again,

$$\Sigma\frac{\cos\psi_1}{r_1^2}\cdot\frac{dr_1}{ds} + \Sigma\frac{\sin\psi_1}{r_1}\cdot\frac{d\psi_1}{ds} = 0, \quad \therefore \quad \Sigma\frac{\cos^2\psi_1}{r_1^2} = -\Sigma\frac{\sin\psi_1}{r_1}\cdot\frac{d\psi_1}{ds};$$

and $$\Sigma\frac{\sin^2\psi_1}{r_1^2} = \Sigma\frac{\sin\psi_1}{r_1}\cdot\frac{d\theta_1}{ds}, \quad \because \ \sin\psi_q = r_q\frac{d\theta_q}{ds};$$

therefore $$\Sigma\frac{1}{r_1^2} = \Sigma\frac{\sin\psi_1}{r_1}\left\{\frac{d\theta_1}{ds} - \frac{d\psi_1}{ds}\right\} = 2\Sigma\frac{\sin^2\psi_1}{r_1^2} - \frac{d\varphi}{ds}\Sigma\frac{\sin\psi_1}{r_1},$$

since $$\varphi = \theta_q + \psi_q;$$

therefore $$\Sigma\left(\frac{1}{mf^2}\right) = 2\Sigma\left(\frac{1}{mp^2}\right) - \frac{1}{m\mu}\Sigma\left(\frac{1}{mp}\right),$$

therefore, &c.

The method here employed may be used with advantage when a vector equation to a curve is given.

7167. (By Professor MINCHIN, M.A.)—A rigid body has uniplanar motion; find the locus, at any instant, of points the directions of whose velocities make a given angle with the directions of their accelerations.

Solution by J. J. WALKER, M.A.

If O is the instantaneous centre, and J the point of no acceleration for any plane section parallel to the given direction of motion, then the direction of acceleration of a point P in the section makes with JP an angle $\phi = \tan^{-1}\omega \div \dot{\omega}^2$ (Quest. 6539, *Reprint*, XXXV., p. 45), which at any instant is the same for all points; while the direction of velocity of P is perpendicular to OP; hence (α being the angle between these directions) the angle $OPJ = \phi + \alpha - \frac{1}{2}\pi$, and is given. The locus of P is therefore a circumference at the points of which the line OJ subtends a given angle; and the locus of points throughout the body satisfying the given condition is a right cylinder.

It is of interest to observe that, as appears from Notes of the late Professor CLIFFORD'S Lectures on "Quaternions" (*Mathematical Papers*, p. 502), delivered in University College, London, in the year 1877, attention was drawn in them to the existence of the point of no acceleration, in the case of uniplanar motion.

6768. (By Professor CROFTON, F.R.S.)—Prove that

$$f(x+h\mathrm{D}).1 \equiv f(x) + \frac{h}{2}f''(x) + \frac{h^2}{2.4}f''''(x) + \&c.$$

Solution by G. F. WALKER, M.A.; Professor SEITZ, M.A.; *and others.*

We have $(x+h\mathrm{D}).1 = x$, $(x+h\mathrm{D})^2.1 = x^2 + \frac{1}{2}h.2$.

Assume $(x+h\mathrm{D})^n.1 = x^n + \ldots \frac{h^r}{2.4\ldots 2r} n(n-1)\ldots(n-2r+1)\,x^{n-2r}$;

then $(x+h\mathrm{D})^{n+1}.1 = \ldots + \frac{h^r}{2.4\ldots 2r} n(n-1)\ldots(n-2r+1)\,x^{n-2r+1}$

$$+ \frac{h^r}{2.4\ldots 2(r-1)} n(n-1)\ldots(n-2r+3)(n-2r+2)\,x^{n-2r+1}$$

$$= \ldots \frac{h^r}{2.4\ldots 2r}(n+1)n(n-1)\ldots(n-2r+2)\,x^{n-2r+1}.$$

Hence, if the formula is true for n, it is true for $n+1$. But it is true for $n=1$ and $n=2$, and therefore generally. Therefore

$$(x+h\mathrm{D})^n.1 = x^n + \ldots \frac{h^r}{2.4\ldots 2r}\frac{d^{2r}}{dx^{2r}}.x^n.$$

But $$f(x) = A_0 + B_1 x + \ldots A_n x^n + \ldots;$$

hence, generally,

$$f(x+h\mathrm{D}).1 = f(x) + \frac{h}{2}f''(x) + \ldots \frac{h^r}{2.4\ldots 2r}f''''(x) + \ldots$$

7101. (By Professor TOWNSEND, F.R.S.)—A cubic surface being supposed to have a nodal point; show that, of the three triple tangent planes intersecting in any one of its twenty-seven superficial lines, the three points of contact not on the line connect with the nodal point by a triad of conjugate diameters of the nodal cone.

Solution by the PROPOSER.

Denoting by O the nodal point, and by I the superficial line of the cubic; by X, Y, Z the three points of contact not on I of the three triple tangent planes to the surface which intersect in I; by P, Q, R the three points of intersection of the three nodal planes YOZ, ZOX, XOY with the line I; and by LL′, MM′, NN′ their three angles of intersection with the nodal cone at O; then, since, for the three nodal sections of the surface by the three aforesaid planes, the three pairs of lines PY and PZ, QZ and QX, RX and RY are three pairs of tangents intersecting at the three points P, Q, R on the three curves, therefore, by a familiar property of plane nodal cubics, the three pairs of lines OY and OZ, OZ and OX, OX and OY divide harmonically the three angles LL′, MM′, NN′, and therefore, &c.

7099. (By the EDITOR.)—Sum (1) the series

$$\frac{\sin^3\theta}{1^4}-\frac{\sin^3 3\theta}{3^4}+\frac{\sin^3 5\theta}{5^4}-\&c.$$

between the limits $\theta=0$, $\theta=2\pi$; and $\theta=\frac{1}{6}\pi$, $\theta=\frac{1}{6}^3\pi$; and hence show (2) that $\pi^4=\frac{3\,.\,2^{4}}{11}\left(\frac{1}{1^4}-\frac{1}{3^4}+\frac{1}{5^4}-\frac{1}{7^4}+\&c.\right)$.

Solution by Prof. WOLSTENHOLME, M.A.; Prof. NASH, M.A.; *and others.*

Transforming each term by the equation $\sin^3 x=\frac{1}{4}(3\sin x-\sin 3x)$, the given series becomes $\frac{3}{4}S_1-\frac{1}{4}S_3$, where

$S_1=\sin\theta-\frac{1}{3}\sin 3\theta+\frac{1}{5}\sin 5\theta-\ldots$, $S_3=\sin 3\theta-\frac{1}{3}\sin 9\theta+\frac{1}{5}\sin 15\theta-\ldots$

Also, for values of ϕ between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$,

$$\tfrac{1}{4}\pi=\cos\phi-\tfrac{1}{3}\cos 3\phi+\tfrac{1}{5}\cos 5\phi-\ldots,$$

and, integrating this three times, we shall have, with the same limits

$$\tfrac{1}{6}\pi\phi\,(\tfrac{1}{4}\pi^2-\tfrac{1}{6}\phi^2)=\sin\phi-\frac{\sin 3\phi}{3^4}+\frac{\sin 5\phi}{5^4}-\frac{\sin 7\phi}{7^4}\quad\ldots\ldots\ldots\ldots(\mathrm{A}).$$

Hence, for a value of ϕ between $r\pi-\frac{1}{2}\pi$ and $r\pi+\frac{1}{2}\pi$, r any integer,

$$(-1)^r\tfrac{1}{6}\pi\,(\phi-r\pi)\,[\tfrac{1}{4}\pi^2-\tfrac{1}{6}(\phi-r\pi)^2]=\sin\phi-\frac{\sin 3\phi}{3^4}+\frac{\sin 5\phi}{5^4}-\frac{\sin 7\phi}{7^4}+\ldots$$

The value of the proposed infinite series will therefore be

(1) from $\theta=-\frac{1}{6}\pi$ to $\theta=\frac{1}{6}\pi$;

$\frac{3}{4}\cdot\frac{1}{6}\tau\theta\,(\frac{1}{4}\pi^2-\frac{1}{6}\tau^2)-\frac{1}{4}\cdot\frac{1}{6}\tau 3\theta\,(\frac{1}{4}\pi^2-3\theta^2)=\frac{1}{4}\pi^3$;

(2) $\theta = \frac{1}{8}\pi$ to $\theta = \frac{1}{4}\pi$;

$$\tfrac{3}{4}\tfrac{1}{8}\pi\,\theta\,(\tfrac{1}{4}\pi^2 - \tfrac{1}{3}\theta^3) + \tfrac{1}{4}\tfrac{1}{8}\pi\,(3\theta-\pi)\,[\tfrac{1}{4}\pi^2 - \tfrac{1}{3}\,(3\theta-\pi)^2];$$

(3) $\theta = \frac{1}{4}\pi$ to $\theta = \frac{3}{8}\pi$;

$$-\tfrac{3}{4}\tfrac{1}{8}\pi\,(\theta-\pi)\,[\tfrac{1}{4}\pi^2 - \tfrac{1}{3}\,(\theta-\pi)^2] - \tfrac{1}{4}\tfrac{1}{8}\pi\,(3\theta-2\pi)\,[\tfrac{1}{4}\pi^2 - \tfrac{1}{3}\,(3\theta-2\pi)^2];$$

(4) $\theta = \frac{3}{8}\pi$ to $\theta = \frac{1}{2}\pi$;

$$-\tfrac{3}{4}\tfrac{1}{8}\pi\,(\theta-\pi)\,[\tfrac{1}{4}\pi^2 - \tfrac{1}{3}\,(\theta-\pi)^2] + \tfrac{1}{4}\tfrac{1}{8}\pi\,(3\theta-3\pi)\,[\tfrac{1}{4}\pi^2 - 3\,(\theta-\pi)^2]$$
$$= -\tfrac{1}{4}\pi\,(\theta-\pi)^3;$$

(5) $\theta = \frac{1}{2}\pi$ to $\theta = \frac{5}{8}\pi$;

$$-\tfrac{3}{4}\tfrac{1}{8}\pi\,(\theta-\pi)\,[\tfrac{1}{4}\pi^2 - \tfrac{1}{3}\,(\theta-\pi)^2] - \tfrac{1}{4}\tfrac{1}{8}\pi\,(3\theta-4\pi)\,[\tfrac{1}{4}\pi^2 - \tfrac{1}{3}\,(3\theta-4\pi)^2];$$

(6) $\theta = \frac{5}{8}\pi$ to $\theta = \frac{11}{16}\pi$;

$$\tfrac{3}{4}\tfrac{1}{8}\pi\,(\theta-2\pi)\,[\tfrac{1}{4}\pi^2 - \tfrac{1}{3}\,(\theta-2\pi)^2] + \tfrac{1}{4}\tfrac{1}{8}\pi\,(3\theta-5\pi)\,[\tfrac{1}{4}\pi^2 - \tfrac{1}{3}\,(3\theta-5\pi)^2];$$

(7) $\theta = \frac{11}{16}\pi$ to $\theta = \frac{13}{16}\pi$;

$$\tfrac{3}{4}\tfrac{1}{8}\pi\,(\theta-2\pi)\,[\tfrac{1}{4}\pi^2 - \tfrac{1}{3}\,(\theta-2\pi)^2] - \tfrac{1}{4}\tfrac{1}{8}\pi\,(3\theta-6\pi)\,[\tfrac{1}{4}\pi^2 - 3\,(\theta-2\pi)^2]$$
$$= \tfrac{1}{4}\pi\,(\theta-2\pi)^3;$$

and, in general, if θ lie between $(2n-1)\,\frac{1}{8}\pi$ and $(2n+1)\,\frac{1}{8}\pi$, the sum will be

$$\tfrac{1}{8}\tfrac{1}{2}\pi\,\big\{(-1)^r\,3\,(\theta - r\pi)\,[\tfrac{1}{4}\pi^2 - \tfrac{1}{3}\,(\theta-r\pi)^2] - (-1)^n\,(3\theta - n\pi)\,[\tfrac{1}{4}\pi^2 - \tfrac{1}{3}\,(3\theta-n\pi)^2]\big\},$$

where r is $\frac{1}{2}\,(n+1)$, or the integer next less.

If, in the equation (A), we put $\phi = \frac{1}{4}\pi$, we get

$$\tfrac{1}{8}\tfrac{1}{2}\pi^4\,(\tfrac{1}{4} - \tfrac{1}{4}\tfrac{1}{8}) = \frac{1}{\sqrt{2}}\left\{\frac{1}{1^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} - \dots\right\},$$

or

$$\pi^4 = \frac{3 \times 2^{\frac{17}{2}}}{11}\left\{\frac{1}{1^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} - \dots\right\};$$

which is also obtained by putting $\theta = \frac{1}{4}\pi$ in the proposed series, and taking the value (2); $\theta = \frac{3}{4}\pi$ and (3); $\theta = \frac{5}{4}\pi$ and (5); or $\theta = \frac{7}{4}\pi$ and (6).

[In the same way, using the identity,

$$\sin^5\phi = \tfrac{1}{16}\,(10\sin\phi - 5\sin 3\phi + \sin 5\phi),$$

we can sum

$$\frac{\sin^5\theta}{1^4} - \frac{\sin^5 3\theta}{3^4} + \frac{\sin^5 5\theta}{5^4} - \frac{\sin^5 7\theta}{7^4} + \dots,$$

and for the first range, from $-\frac{1}{10}\pi$ to $\frac{1}{10}\pi$, the sum is zero; and will, in general, be

$$\tfrac{1}{1}\tfrac{1}{2}\tfrac{1}{8}\pi\,(10S_1 - 5S_3 + S_5),$$

where

$$S_1 = (-1)^p\,(\theta - p\pi)\,[\tfrac{1}{4}\pi^2 - \tfrac{1}{3}\,(\theta-p\pi)^2],$$
$$S_3 = (-1)^q\,(3\theta - q\pi)\,[\tfrac{1}{4}\pi^2 - \tfrac{1}{3}\,(3\theta-q\pi)^2],$$
$$S_5 = (-1)^r\,(5\theta - r\pi)\,[\tfrac{1}{4}\pi^2 - \tfrac{1}{3}\,(5\theta-r\pi)^2];$$

where p, q, r are integers such that each of the angles $\theta - p\pi$, $3\theta - q\pi$, $5\theta - r\pi$ lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. It is obvious that, in the same way, the sum of the infinite series

$$\frac{\sin^n\left(\frac{\theta}{n}\right)}{1^{2m}} - \frac{\sin^n\left(\frac{3\theta}{n}\right)}{3^{2m}} + \frac{\sin^n\left(\frac{5\theta}{n}\right)}{5^{2m}} - \dots,$$

where n is any odd integer, and m an integer less than $\frac{1}{2}n$, will be zero from $\theta = -\frac{1}{2}\pi$ to $\theta = \frac{1}{2}\pi$; and

$$\frac{\sin^{2m-1}\theta}{1^{2m}} - \frac{\sin^{2m-1}(3\theta)}{3^{2m}} + - \dots = \tfrac{1}{4}\pi\,\theta^{2m-1},$$

from $\theta = -\dfrac{\pi}{4m-2}$ to $\theta = \dfrac{\pi}{4m-2}$.]

7066. (By Dr. MACFARLANE, F.R.S.E.)—Find the number of ways in which an elector may vote, under the cumulative system of voting, when there are p candidates for q seats. [Applied to the recent election of a School Board for Dundee, Dr. MACFARLANE'S results show that the number of possible ways of voting was 43 times the population of the globe.]

Solution by the PROPOSER.

Under the cumulative system of voting, an elector may give any number r of votes not greater than q, the number of seats; and he may distribute the r votes in any manner he pleases among the p candidates. The number of ways in which he can distribute r votes among p candidates is the same as the number of homogeneous products of r dimensions which can be formed out of p symbols; hence it is $\frac{p+r-1\,!}{p-1\,!\;r\,!}$. Hence the total number of ways in which the elector may vote is $\frac{p+q!}{p!\;q!}$; and, as any mathematician will at once see, the electors will, in general, have an ample choice.

7047. (By R. RUSSELL, B.A.)—Show (1) generally how to solve

$$(a_1 b_1 c_1 f_1 g_1 h_1)(x, y, z)^2 = l_1, \quad (a_2 b_2 c_2 f_2 g_2 h_2)(x, y, z)^2 = l_2,$$
$$(a_3 b_3 c_3 f_3 g_3 h_3)(x, y, z)^2 = l_3;$$

and (2) prove that the solution of

$$x^2 + yz = a, \quad y^2 + zx = b, \quad z^2 + xy = c,$$

depends on the solution of $u^3 + v^3 + w^3 = 5uvw$, where

$$u = (b-c)\left(1+\frac{\lambda}{a}\right), \quad v = (c-a)\left(1+\frac{\lambda}{b}\right), \quad w = (a-b)\left(1+\frac{\lambda}{c}\right).$$

[This is connected with the EDITOR'S Questions 6928 and 6972.]

Solution by the PROPOSER; D. EASTWOOD, M.A.; *and others.*

$$S_1 \equiv (a_1 b_1 c_1 f_1 g_1 h_1)(xyz)^2 = l_1, \quad S_2 \equiv (a_2 b_2 c_2 f_2 g_2 h_2)(xyz)^2 = l_2,$$
$$S_3 \equiv (a_3 b_3 c_3 f_3 g_3 h_3)(xyz)^2 = l_3;$$

therefore
$$U \equiv (l_2 - l_3) S_1 + (l_3 - l_1) S_2 + (l_1 - l_2) S_3 = 0,$$
$$V \equiv \frac{l_2 - l_3}{l_1} S_1 + \frac{l_3 - l_1}{l_2} S_2 + \frac{l_1 - l_2}{l_3} S_3 = 0.$$

The determination of λ such that $U + \lambda V$ may represent two linear factors depends on a cubic of course.

Hence, in solving this cubic, we obtain a line as relation between x, y, z. By means of which, and any two of the original equations, we can at once solve for x, y, and z.

In the particular case of $x^2+yz=a$, $y^2+zx=b$, $z^2+xy=c$,

$$U+\lambda V \equiv (a^2+yz)(b-c)\left(1+\frac{\lambda}{a}\right)+(y^2+zx)(c-a)\left(1+\frac{\lambda}{b}\right)$$
$$+(z^2+xy)(a-b)\left(1+\frac{\lambda}{c}\right)=0,$$

or $ux^2+vy^2+wz^2+uyz+vzx+wxy=0$, therefore $u^3+v^3+w^3=3uvw$,

where $u=(b-c)\left(1+\frac{\lambda}{a}\right)$, $v=(c-a)\left(1+\frac{\lambda}{b}\right)$, $w=(a-b)\left(1+\frac{\lambda}{c}\right)$.

7125. (By Professor WOLSTENHOLME, M.A.)—Prove that, if

$$y=\frac{1}{\epsilon^x-1}, \quad \frac{d^ny}{dx^n}=(-1)^n\left\{\frac{A_1}{\epsilon^x-1}+\frac{A_2}{(\epsilon^x-1)^2}+\ldots+\frac{A_{n+1}}{(\epsilon^x-1)^{n+1}}\right\},$$

where $$A_p=\Delta^p 0^n+\Delta^{p-1}0^n.$$

Solution by W. J. C. SHARP, M.A.; Professor MATZ, M.A.; *and others.*

If $y=(\epsilon^x-1)^{-1}$, since this may be written

$$\epsilon^{-x}+\epsilon^{-2x}+\epsilon^{-3x}+\ldots+\epsilon^{-rx}+\ldots,$$

we have $$\frac{d^ny}{dx^n}=(-1)^n(\epsilon^{-x}+2^n\epsilon^{-2x}+3^n\epsilon^{-3x}+\ldots+r^n\epsilon^{-rx}+\ldots).$$

Now $$\frac{1}{z}+\frac{2^n}{z^2}+\frac{3^n}{z^3}+\ldots \text{ to } \infty \equiv \frac{A_1}{z-1}+\frac{A_2}{(z-1)^2}+\ldots+\frac{A_{n+1}}{(z-1)^{n+1}},$$

if $A_1=1^n$, $A_1+A_2=2^n$, $A_1+2A_2+A_3=3^n$, &c.; and these equations may be written $A_1=1^n$, $A_1+A_2=(1+\Delta)1^n$, $A_1+2A_2+A_3=(1+\Delta)^2 1^n$, and so on, from which form it is obvious that

$$A_p=\Delta^{p-1}1^n=\Delta^{p-1}0^n+\Delta^p 0^n.$$

[It is easy to prove that

$$A_1+A_3+A_5+\ldots=A_2+A_4+A_6+\ldots$$
$$=2^nA_{n+1}-2^{n-1}A_n+2^{n-2}A_{n-1}-\ldots+(-1)^nA_1,$$
$$=\tfrac{1}{2}\left(1^n+\frac{2^n}{2}+\frac{3^n}{2^2}+\ldots+\frac{r^n}{2^{r-1}}+\ldots \text{ to } \infty\right),$$

the last series being always convergent.]

7084. (By J. GRIFFITHS, M.A.)—Find the locus of intersection of three mutually perpendicular planes to an elliptic paraboloid.

Solution by D. EDWARDES; J. O'REGAN; *and others.*

Let the paraboloid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$, and a tangent plane thereto

$l_1x+m_1y+n_1z=p\equiv-\frac{a^2l_1^2+b^2m_1^2}{2cn_1}$, or $n_1l_1x+m_1n_1y+n_1^2z+a^2l_1^2+b^2m_1^2=0$;

whence, since $\Sigma n_1l_1=0$, $\Sigma m_1n_1=0$, $\Sigma n_1^2=\Sigma l_1^2=\Sigma m_1^2=1$, we have, by addition for the required locus, the plane $z+\frac{a^2+b^2}{2c}=0$.

7179. (By the Rev. G. RICHARDSON, M.A.)—If AOD be the perpendicular from A on the opposite side of the triangle ABC, O being the orthocentre, I the centre of the inscribed circle, r its radius; prove that $IO^2=2r^2-AO\,.\,OD$.

Solution by W. H. BLYTHE, M.A.; R. KNOWLES, B.A.; *and others.*

Here we readily find $AD=c\sin B$, $OD=b\cos C\,.\,\cot B$,

therefore $AO=AD-OD=b\cos A\operatorname{cosec}B$;

$\therefore\ AO\,.\,OD=b^2\operatorname{cosec}^2B\,.\cos A\cos B\cos C=4R^2\cos A\cos B\cos C$;

also (see solution of Quest. 6313),

$$OI^2=2r^2-4R^2\cos A\cos B\cos C,$$

therefore $=2r^2-AO\,.\,OD$.

7188. (By Dr. CURTIS.)—Prove that the centre of gravity of the area included by a loop of the curve $r=a\cos n\theta$ is at a distance from the origin

$$=16an^4\sin\left(\frac{n+1}{2n}\pi\right)\div\{(n^2-1)[(3n)^2-1]\pi\}.$$

[The cases $n=2$, $n=3$ are exercises in TODHUNTER'S *Statics.*]

Solution by W. M. COATES, B.A.; W. H. BLYTHE, B.A.; *and others.*

In the curve $r=a\cos n\theta$, $r=a$ for $\theta=0$ and $r=0$ for $\theta=\frac{\pi}{2n}$, and as loop is symmetrical with respect to line $r=a$, if $\bar{x}$ denote distance of centre of gravity of loop from origin, $\bar{x}=\frac{\frac{1}{3}\int r^3\cos\theta\,d\theta}{\int r^2\,d\theta}$, the integrations

being taken between $\theta = \frac{\pi}{2n}$ and $\theta = -\frac{\pi}{2n}$. But

$$\int r^3 \cos\theta\, d\theta = a^3 \int \cos^3 n\theta \cos\theta\, d\theta = \tfrac{1}{4}a^3 \int (\cos 3n\theta + 3\cos n\theta)\cos\theta\, d\theta$$

$$= \tfrac{1}{8}a^3 \int [\cos(3n+1)\theta + \cos(3n-1)\theta + 3\cos(n+1)\theta + 3\cos(n-1)\theta]\, d\theta$$

$$= \tfrac{1}{8}a^3 \left[\frac{\sin(3n+1)\theta}{3n+1} + \frac{\sin(3n-1)\theta}{3n-1} + \frac{3\sin(n+1)\theta}{n+1} + \frac{3\sin(n-1)\theta}{n-1} \right]_{-\frac{\pi}{2n}}^{\frac{\pi}{2n}}$$

$$= \tfrac{1}{4}a^3 \cdot \frac{8n^2}{(n^2-1)[(3n)^2-1]} \cdot 6n \sin\left(\frac{n+1}{2n}\right)\pi, \text{ between the limits.}$$

Again, $\int r^2\, d\theta = a^2 \int \cos^2 n\theta\, d\theta =$ (between same limits) $\frac{a^2 \cdot \pi}{2n}$,

$$\therefore \bar{x} = \frac{\frac{3}{8} \cdot \frac{1}{4}a^4 \cdot \frac{8n^2}{(n^2-1)[(3n)^2-1]} \cdot 6n \sin\left(\frac{n+1}{2n}\right)\pi}{\frac{a^2\pi}{2n}} = \frac{16an^4 \sin\left(\frac{n+1}{2n}\pi\right)}{(n^2-1)[(3n)^2-1]\pi}.$$

7181. (By H. L. Orchard, M.A.)—Given that the sum of the radii of the inscribed, escribed, and circumscribed circles of a certain equilateral triangle is unity, find the area of the triangle.

Solution by C. Morgan, M.A.; R. Knowles, B.A.; *and others.*

If x = side of equilateral triangle, we have

$$R = \tfrac{1}{3}\sqrt{3} \cdot x, \quad r = \tfrac{1}{6}\sqrt{3} \cdot x, \quad r_a + r_b + r_c = \tfrac{3}{2}\sqrt{3} \cdot x;$$

and, equating the sum of these to unity, we get

$$x = \tfrac{1}{6}\sqrt{3}, \quad \text{and} \quad \text{area} = \tfrac{1}{4\,3}\sqrt{3}.$$

7175. (By G. G. Morrice, B.A.)—Prove that the angle between any two straight lines in space $= \sin^{-1} | l_1, m_2, n_3 |$, where $l_1 m_1 n_1$, $l_2 m_2 n_2$ are their direction-cosines, and $l_3 m_3 n_3$ those of their shortest distance.

Solution by G. Turriff, M.A.; W. M. Mee, B.A.; *and others.*

$$\frac{l_3}{m_1 n_2 - m_2 n_1} = \frac{m_3}{n_1 l_2 - n_2 l_1} = \frac{n_3}{l_1 m_2 - l_2 m_1} = \pm \frac{1}{\sin\theta}$$

(Frost's *Solid Geometry*, Art. 23),

therefore $\sin\theta = \pm [l_3(m_1 n_2 - m_2 n_1) + m_3(n_1 l_2 - n_2 l_1) + n_3(l_1 m_2 - l_2 m_1)]$

$$= \pm \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}.$$

7171. (By E. W. Symons, M.A.)—Prove that, in any triangle,
$$\cot A + \cot^2 B + \cot^2 C > 1.$$

Solution by H. L. Orchard, M.A.; J. S. Jenkins, M.A.; *and others.*

$\sin^2 A \sin^2 B \cos^2 C + \sin^2 B \sin^2 C \cos^2 A + \sin^2 C \sin^2 A \cos 2B$ is > 0,

i.e., $\sin^2 A \sin^2 B \cos^2 C + \sin^2 B \sin^2 C \cos^2 A + \sin^2 C \sin^2 A (\cos^2 B - \sin^2 B) > 0$,

$\sin^2 A \sin^2 B \cos^2 C + \sin^2 B \sin^2 C \cos^2 A + \sin^2 C \sin^2 A \cos^2 B > \sin^2 A \sin^2 B \sin^2 C$,

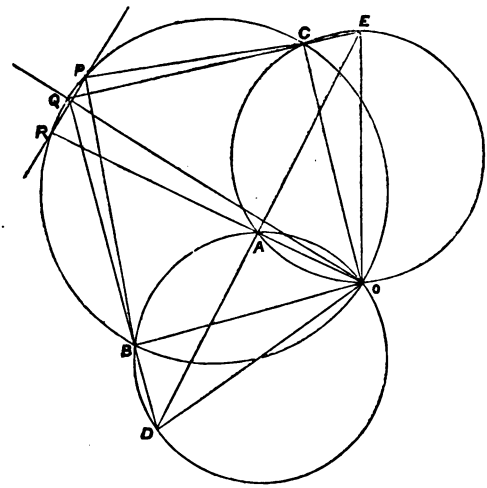
and hence $\cot^2 A + \cot^2 B + \cot^2 C > 1$.

7112. (By R. F. Scott, M.A.)—Three circles which pass through the same point O, intersect again in A, B, C. Through the point A a line is drawn meeting the circles which intersect at A in D and E. The straight lines DB, EC meet the third circle again in F and G. Prove that, as the line DAE varies, the straight line FG envelopes a circle concentric with the circle BOC.

Solution by J. W. Sharpe, M.A.; E. Rutter; *and others.*

Let the lines DB and EC meet at P; then

$\angle PDA$ or $\angle BDA = \angle AOB$, also $\angle PEA$ or $\angle CEA = \angle AOC$,



therefore $\angle PDA + \angle PEA = \angle BOC$;

therefore $\angle BPC$ and $\angle BOC$ are supplementary; and therefore P lies on the third circle. Hence the points F and G are coincident.

The same result may be otherwise obtained analytically, and in the course of the investigation it may easily be shown that the lines DBP, ECP are equally inclined to BQ, CQ respectively, where OQ is a diameter of the third circle, and that the angles they make with these lines are complementary to that which DAE makes with OA, as is geometrically evident; furthermore, that PR is parallel to DAE, which also is easily proved geometrically. Thus we have the following theorem:—If the radical axis OA of any two of the circles be produced to cut the third circle in the point R, and the line RP be thus determined, then the three lines such as RP form a triangle whose sides are parallel respectively to those of PDE.

7115. (By A. McMurchy, B.A.)—If $a_1, a_2, a_3 \ldots$ be the lengths of the sides of a convex polygon ABCD... inscribed in a circle; $p_1, p_2, p_3 \ldots$ the lengths of the perpendiculars on the considered position from any point P on the smaller arc cut off by a_1; prove that

$$\frac{a_1}{p_1} = \frac{a_2}{p_2} + \frac{a_3}{p_3} + \ldots + \frac{a_n}{p_n}.$$

Solution by J. O'Regan; J. W. Sharpe, M.A.; *and others.*

Let ABC be a triangle inscribed in a circle, P being on the arc AB; then $\Delta PP_2P_1 + \Delta PP_1P_3 = \Delta PP_2P_3$, *i.e.*, $p_1p_2 \sin B + p_1p_3 \sin A = p_2p_3 \sin C$; but $\frac{\sin B}{a_3} = \frac{\sin A}{a_2} = \frac{\sin C}{a_1}$, therefore $\frac{a_1}{p_1} = \frac{a_2}{p_2} + \frac{a_3}{p_3}$.

Now, in case of a polygon, draw the line AC, and let q be the perpendicular thereon; then

$$\frac{a_1}{p_1} = \frac{a_2}{p_2} \frac{b_1}{q_1}, \quad \frac{b_1}{q_1} = \frac{a_3}{p_3} + \frac{b_2}{q_2}, \ \ldots, \quad \frac{b_{n-3}}{q_{n-3}} = \frac{a_{n-1}}{p_{n-1}} - \frac{a_n}{p_n}; \quad \therefore \frac{a_1}{p_1} = \frac{a_2}{p_2} + \ldots + \frac{a^n}{p_n}.$$

7213. (By C. Morgan, B.A.)—Show that, in any triangle,

$$s \cdot \Sigma(r_a^{-1}) = \cot \tfrac{1}{2}A \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C,$$

where (as usual) r_a is the radius of the circle escribed to the side a, and s is the semiperimeter.

Solution by R. Knowles, B.A.; G. Turriff, M.A.; *and others.*

The first member $= \Delta^{-1} s (s_1 + s_2 + s_3)$, where $s_1 = s - a$, &c. $= \Delta^{-1} s^2$; the second $= s^{\frac{3}{2}} (s_1 s_2 s_3)^{-\frac{1}{2}} = \Delta^{-1} s^2$.

5689. (By Professor CAYLEY, F.R.S.)—Show (1) that the apparent contour of a Steiner's surface $(2xyz + y^2z^2 + z^2x^2 + x^2y^2 = 0)$, as seen from an exterior point on a nodal line (say the axis of z), projected on the plane of the other two nodal lines, is an ellipse passing through the four points $(\pm 1, 0)$ and $(0, \pm 1)$; and (2) find the surface-contour, or curve of contact of the cone and surface.

Solution by W. J. C. SHARP, M.A.; Professor MATZ, M.A.; *and others.*

The equation to the tangent plane to the Steiner's surface at (x, y, z) may be put in the form $\frac{yz}{x}(yz+x)\xi + \frac{zx}{y}(zx+y)\eta + \frac{xy}{z}(xy+z)\zeta = xyz$, where ξ, η, ζ are the current coordinates.

If this plane cut the nodal line (axis of z) at a distance a from the origin, $(xy+z)a = z^2$, and from this equation combined with $y^2z^2 + z^2x^2 + x^2y^2 + 2xyz = 0$ it follows that $x^2 + y^2 + \frac{(z-a)^2}{a^2} + 2\frac{z-a}{a} = 0$, that is, $x^2 + y^2 + \frac{z^2}{a^2} - 1 = 0$, a quadric of revolution on which the curve of contact lies.

The equations to the line joining $(0, 0, a)$ to (x, y, z), the point of contact, are $\frac{\xi}{x} = \frac{\eta}{y} = \frac{\zeta - a}{z - a}$, and this is the generator of the enveloping cone; at the points where this meets the plane of xy, $\zeta = 0$,

and therefore $$x = -\frac{\xi(a-z)}{a}, \quad y = -\frac{\eta(z-a)}{a};$$

and therefore $$xy = \frac{\xi\eta}{a^2}(z-a)^2 = \frac{z(z-a)}{a}, \text{ from above;}$$

and therefore $$\frac{z-a}{z} = \frac{a}{\xi\eta}; \text{ and therefore } \frac{z-a}{a} = \frac{a}{\xi\eta - a};$$

$\therefore$ since $$x^2 + y^2 + \frac{(z-a)^2}{a^2} + 2\frac{z-a}{a} = 0, \quad (\xi^2 + \eta^2 + 1)\frac{a^2}{(\xi\eta - a)^2} + 2\frac{a}{\xi\eta - a} = 0.$$

Therefore $(\xi^2 + \eta^2)a + 2\xi\eta - a = 0$ is the equation to the apparent contour an ellipse through the four points.

6979. (By Professor TOWNSEND, F.R.S.)—A spiral intersecting at any constant angle all the meridians of any convex surface of revolution, being supposed the frictionless catenary of a uniform cord, or the frictionless trajectory of a material particle, constrained to rest or move on the surface, under the action of a force passing perpendicularly in every position through the axis of revolution; show that the force varies, for the catenary inversely as the square, and for the trajectory inversely as the cube, of the distance from the axis.

Solution by the PROPOSER.

Denoting by ρ and ψ the axial distance and azimuthal angle of any point on the spiral, by T and v the tension of the cord and the velocity of the

particle at the point, and by k the cosecant of the constant angle of intersection of the curve with all the meridians of the surface; then since, on elementary dynamical principles, $d\Gamma = -F d\rho$, and $T\rho^2 d\psi = h ds$, where h is a constant, for the catenary, and, $v dv = F d\rho$, and $v\rho^2 d\psi = h ds$, where h again is a constant, for the trajectory, and since, from the fundamental property of the spiral, $ds = k\rho d\psi$ for both cases alike, therefore at once, $F = hk \cdot \rho^{-2}$ for the catenary, and $= -h^2k^2 \cdot \rho^{-3}$ for the trajectory; and therefore, &c.

7033. (By Professor HUDSON, M.A.)—A particle is attracted to one and repelled from another centre of force varying inversely as the square of the distance and of equal absolute intensity; show that the surfaces of equilibrium are those formed by the revolution of parts of the curves

$$(r^2+a^2)^2(r^2+a^2-2c^2)+4a^2r^2\cos^2\theta\,[c^4-2(r^2+a^2)(r^2+a^2-c^2)], \\ +16a^4r^4\cos^4\theta = 0$$

where $2a$ is the distance between the centres of force. What do the other parts of these curves represent?

Solution by W. H. BLYTHE, M.A.; J. HAMMOND, M.A.; *and others.*

If r_1, r_2 be the distances of the particle from the centres of force respectively, the surfaces of equilibrium are given by

$$\int\left(\frac{\mu\,dr_1}{r_1^2}-\frac{\mu\,dr_2}{r_2^2}\right) = \text{constant}\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let $2a$ be the distance between the centres, take the middle point as origin polar coordinates, then

$$r_1^2 = r^2+a^2-2ar\cos\theta \equiv t-p \text{ (suppose)},$$
$$r_2^2 = r^2+a^2+2ar\cos\theta \equiv t+p \text{ (suppose)}.$$

Integrate (1) and square; then, the constant being arbitrary, we can take as solution $c^2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)^2 = 2$; and, substituting the values for r_1 and r_2,

$$c^2[(t+p)-2(t+p)^{\frac{1}{2}}(t-p)^{\frac{1}{2}}+(t-p)] = 2(t^2-p^2),$$

or
$$c^2(t^2-p^2)^{\frac{1}{2}} = t^2-p^2-c^2t.$$

Squaring, $c^4(t^2-p^2) = t^4+p^4+c^4t^2-2t^2p^2-2c^2t^3+2c^2p^2t\ldots\ldots\ldots\ldots\ldots(2)$;

therefore
$$t^3(t-2c^2)+p^2[c^4-2t(t-c^2)]+p^4 = 0, \text{ or}$$

$$(r^2+a^2)^2(r^2+a^2-2c^2)+4a^2r^2\cos^2\theta\,[c^4-2(r^2+a^2)(r^2+a^2-c^2)]+16a^4r^4\cos^4\theta.$$

It is to be noticed that equation (2) is obtained also by simplifying $c^2\left(\frac{1}{r_1}+\frac{1}{r_2}\right)^2 = 2$, and thus includes also the surfaces of equilibrium when both forces are attractive or both repulsive.

4667. (By the late Professor CLIFFORD, F.R.S.)—If a conic be inverted into a circular cubic with a double point, prove that the foci and directrices of the conic will invert into foci and directing circles of the cubic.

Solution by W. J. C. SHARP, M.A.; G. HEPPEL, M.A.; *and others.*

If $(x-h)^2+(y-k)^2 = e^2(x-m^2p)^2$ be the conic, so that (h, k) is the focus, e the excentricity, and $x-m^2p = 0$ the directrix; and if m be the radius of inversion, $(x-h)^2+(y-k)^2$ becomes

$$\frac{m^4}{X^2+Y^2} - \frac{2(HX+KY)m^4}{(X^2+Y^2)(H^2+K^2)} + \frac{m^4}{H^2+K^2},$$

where (X, Y) and (H, K) are the points inverse to (x, y) and (h, k),

or $$\frac{m^4}{(X^2+Y^2)(H^2+K^2)}[(X-H)^2-(Y-K)^2];$$

and the equation to the inverse is

$$(X-H)^2+(Y-K)^2 = \frac{e^2(H^2+K^2)}{X^2+Y^2}[X-p(X^2+Y^2)]^2.$$

So that H, K is a focus [the intersection of the two tangents $(X-H)\pm\sqrt{-1}(Y-K)$ through the circular points], and the circle $X-p(X^2+Y^2)$ passes through the points where they meet the curve, and is a directing circle. The theorem, as proved, applies wherever the centre of inversion may be, and therefore whether the inverse be a bicircular quartic or a circular cubic. It is, indeed, more general, and may be enunciated as follows:—If any curve be inverted, the foci and directing curves will invert into foci and directing curves of the inverse curve.

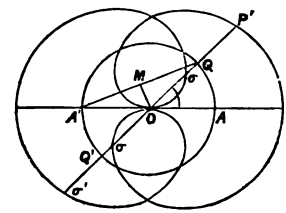
5529. (By the late T. COTTERILL, M.A.)—The curve $r = a\sin\frac{1}{2}(\lambda\pi+\theta)$, where λ is an integer, being a pedal of a bi-cusped epicycloid, is unicursal; find (1) the positions of its singular points and of its four real bi-tangents; and (2) give a simple construction for tracing the curve.

NOTE *by the* EDITOR.

Among the very last articles that our valued correspondent Mr. COTTERILL sent was the annexed diagram of his Quest. 5529, where, as stated in an accompanying letter, "the radius vector

$O\sigma$ = chord AQ = 2OM."

Mr. COTTERILL remarks that "the curve has been traced mechanically by Mr. PERIGAL, but it has many remarkable properties besides those mentioned in the Question. The figure explains what I mean."



4971. (By Professor SYLVESTER, F.R.S.)—The equation to the inverse of a conic in Cartesian coordinates being given, find the algebraical expression for the node in its simplest terms.

Solution by W. J. C. SHARP, M.A.; ELIZABETH BLACKWOOD; *and others.*

If $ax^2 + 2hxy + by^2 + 2fy + 2gx + c = 0$ be the conic referred to any rectangular axes, and m the radius of inversion, the inverse is

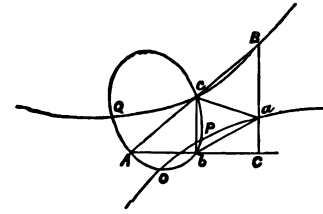
$$m^4(aX^2 + 2hXY + bY^2) + 2(fY + gX)m^2(X^2 + Y^2) + c(X^2 + Y^2)^2 = 0,$$

and therefore the nodal tangents at the origin are parallel to the asymptotes to the conic, and the node is a crunode, a cusp, or an acnode, according as $h^2 - ab$ is positive, zero, or negative. More generally, the inverse of an n-ic, which does not pass through the circular points, has an n-ic point at the origin, the tangents at which are parallel to the asymptotes of the curve inverted.

7075. (By Professor WOLSTENHOLME, M.A.)—If in a triangle ABC there be inscribed another triangle abc, and two conics through Bca, Cab, intersect again in A′, B′, C′; prove that the conic inscribed in ABC and touching two sides (A′B′, A′C′) of the triangle A′B′C′ will have double contact with the conic through the five points a, b, c, B′, C′.

Solution by the PROPOSER.

Given a triangle ABC and a conic U, meeting the sides of the triangle in the points a, a'; b, b'; c, c' respectively; then four conics can be inscribed in the triangle which shall have double contact with the conic. One of them is determined by the following construction: draw a conic V_1 through Abc meeting U again in P, Q, the conic V_2 through Bca PQ (as also the conic V_3 through Cab PQ) will meet V_1 in a fourth point O, and the (unique) conic inscribed in ABC and touching OP, OQ will have double contact with U. The same conic might also be determined by taking $a'b'c'$ in place of abc; and the other three will be found in the same way from the groups of points $a'bc$ or $ab'c'$; $ab'c$ or $a'bc'$; abc' or $a'b'c$. These four conics will of course always be real when the points abc $a'b'c'$ are real, and the chord of contact will then also be real, but the points of contact may be impossible.



The property, assumed in the construction, that the conics $V_1V_2V_3$ have a common point O (the projection of a well known property of circles), is readily proved from the fundamental anharmonic point property as follows: Let abc be a triangle inscribed in ABC, V_1 any conic through Abc, V_2 a conic through Bca meeting V_1 also in OPQ, then if we consider the six points Cab OPQ, we have

$$a\,|\,COPQ\,| = a\,|\,BOPQ\,| = c\,|\,BOPQ\,|$$
$$= c\,|\,AOPQ\,| = b\,[AOPQ] = b\,[COPQ];$$

hence $b \mid COPQ \mid = a \mid COPQ \mid$, or the six points abC OPQ lie on one conic.

This is of course equivalent to the theorem that the circles Abc, Bca, Cab, concur in a point O. Then I further assert that the (unique) conic touching OP, OQ and the sides of the triangle ABC will have double contact with the conic through abc PQ.

6932. (By the Rev. T. P. KIRKMAN, M.A., F.R.S.)—It is an old theorem that every $(12+r)$-edron P, whose summits are all triedral, and whose faces are all 5-gons and 6-gons, has twelve pentagons. Required the number of these solids (P) for values of r up to $r=7$, no one being counted which can be drawn in any position either as the repetition or the reflected image of another; together with a complete account of symmetry, enumerating the symmetrical faces, summits, and edges. The required number is less than 20.

Solution by the PROPOSER.

The solids required are—for $r=0$, one zoned hexarchaxine, whose six principal axes have 5-zoned 5-gons, whose ten secondary axes have 3-zoned triaces, and whose fifteen tertiary axes have 2-zoned edges (55), for poles. The 31 axes are all janal.

For $r=2$, one 6-zoned monarchaxine homozone, whose zoned principal axis has 6-gons, and whose six zoneless secondary axes have twelve edges (55), for zoneless poles. All axes janal.

For $r=3$, A_3, a 3-zoned monarchaxine heterozone, has 3-zoned triaces for poles of the principal axis, while its three like secondary heteroid (= non-janal) axes have each for 2-zoned poles a 6-gon and an edge 55. (A monarchaxine homozone has only one zoned axis and not two different zones; a monarchaxine heterozone has more than one zoned axis and has different zones.)

For $r=4$, A_4, a zoned tetrarchaxine, has for poles terminating each of its four principal 3-zoned heteroid axes, a 3-zoned 6-gon and triace; while its three secondary janal 2-zoned axes carry one zoned polar edge (55). Also B_4 is a zoneless triaxine, whose three different 2-ple janal axes have, one poles (66), a second poles (55), and the third other poles (55), all zoneless.

For $r=5$; A_5 is a 2-zoned monaxine heteroid, whose poles are a 2-zoned 6-gon and an edge (55). B_5 is another 2-zoned monaxine heteroid, whose axis carries a zoned polar edge (66) and a 2-zoned 6-gon. A third solid is C_5, a 5-zoned monarchaxine heterozone, on which a 5-gon terminates the principal janal axis, while the five secondary heteroid 2-zoned axes have each for poles a 6-gon and an edge (66).

For $r=6$; A_6 is a 3-zoned monarchaxine heterozone: the poles of the janal principal axis are 3-zoned triaces, and the three 2-zoned heteroid secondaries have for poles each two different edges (55). Next B_6 is a 3-ple zoneless monarchaxine. The two principal janal poles are the same 3-ple triace, and the three heteroid secondary axes have each for poles (66) and (55). A third solid is C_6, a 2-ple zoneless monaxine heteroid, whose poles are two different (55). Similar to C_6 is D_6, whose 2-ple poles are (66) and (55). Next E_6 is a 3-zoned monarchaxine homozone, whose

principal janal poles are 3-zoned triaces, its six janal secondary poles of three zoneless axes being the same zoneless polar (66). A sixth solid is F_6, a zoneless triaxine, whose six janal 2-ple poles are a 6-gon, an edge (66) and an edge (55), on three different janal axes.

For $r = 7$; A_7 is a 3-zoned monaxine heteroid on whose axis the 3-zoned poles are a triace and a 6-gon. B_7 is a 2-ple zoneless monaxine heteroid whose poles are a 2-ple 6-gon and an edge (55). C_7 is a mononone, whose zoned faces are four 5-gons, and one 6-gon with diagonal trace. It has three zonal and two epizonal edges. D_7 is a monozone, having 2 zoned 5-gons, one 6-gon with agonal trace and two 6-gons diagonally traced. It has also three zonal and two epizonal edges. E_7 and F_7 are 2-ple monaxine heteroids, of which the first has for zoneless poles a 2-ple 6-gon and edge (66) and the second, like B_7, a 6-gon and (55).

These nineteen, one 14-edron, one 15-edron, two 16-edra, three 17-edra, six 18-edra, and six 19-edra, are the solids required for $r > 0$, up to $r = 7$.

It would be unbecoming not to remark, before I give my way (it will come better with the solution of Question 6964) of finding and construction on those for lower values of these results, that Professor Cayley has invented (Memoirs of Literary and Philosophical Society of Manchester, Third Series, Vol. I., 1862, p. 248) a method of constructing from triangular-faced *data*, the P-acral Q-edra whose faces are all triangles; or, what is the same thing, the P-edral Q-acra whose summits are all triaces, on data with like summits. The most important thing in his paper is a special method of solving the questions before us here, and any extension of them.

This remark would be called for, if the writer were even far below the stature of a Cayley, on the only attempt in any language that I have heard of, besides my own humble efforts, to touch the Polyedra.

Professor Cayley has made of this special method no application whatever, of which I am aware, neither have I. The only results given in his paper are the enumeration and construction of the 8-acra whose faces are triangular, with triaces and tessaraces among the summits, that is of the 8-edra whose summits are tri-edral, without exclusion of triangles and 4-gons. This he does evidently unaware at the time that I had enumerated and *constructed* the same solids in the same Memoirs, Vol. XII., 1854. My triplets are neither more nor less than his triangles. But one defect runs through both Professor Cayley's paper and my earlier one, in that no correct account is given of symmetry. Nothing would give me more pleasure, in connexion with Polyedra, than to see Professor Cayley's method applied to obtain the results above given, together with a few more as far as $r = 10$ or $= 12$. I am persuaded that, backed by sufficient industry and very skilful ordering of data and results, the method will do the work, if only the worker be master of the grand puzzle of polyedral symmetry. For information on that head, I can only refer to a chapter, No. XXV., of the Philosophical Transactions for 1862, in which will be found definitions and explanations of the terms above employed. There is much to be discovered in choosing out of a number the most elegant method of constructing detached sub-families of polyedra, so as to preclude omissions and repetitions. In the general theory, the enumeration of the triangular-faced solids from those of one edge fewer is, beyond all comparison, the simplest and briefest task in this study that can be set. When the enumeration and description are completed, the construction of the solids is a bagatelle, with the tables of edges all before us. For an example, see the "Proceedings of the Literary and Philosophical Society of Liverpool," Vol. XXXII., 1877-8.

1100. (By the Editor.)—If a straight line be divided at random into three parts, show that the chance (1) that a triangle of any kind can be formed with the three parts is $\frac{1}{4}$; and (2) that an acute-angled triangle can be formed by the three parts is $3\log 2 - 2$.

Solution by W. T. A. Emtage; Prof. Matz, M.A.; *and others.*

1. Let x, $y-x$, $2s-y$ be the three parts; then the required chance is

$$\iint dy\,dx \div \iint dy\,dx,$$

where the limits in the numerator are given by $x < s$, $y > s$, $y - x < s$; and in the denominator x and y have all values possible, y being $> x$.

Thus the required chance becomes

$$\int_s^{2s}\int_{y-s}^{s} dy\,dx \div \int_0^{2s}\int_0^{y} dy\,dx = \int_s^{2s}(2s-y)\,dy \div \int^{2s} y\,dy = \frac{2s^2 - \frac{1}{2}(4s^2 - s^2)}{\frac{4}{3}s^2} = \frac{1}{4}.$$

2. To find the chance that a triangle is formed, and that the segment x subtends an obtuse angle, we have the condition

$$x^2 > (y-x)^2 + (2s-y)^2, \quad \therefore\ 0 > 2y^2 - 2xy + 4s^2 - 4sy, \quad \therefore\ x > y + \frac{2s^2}{y} - 2s;$$

hence the numerator of the fraction giving the chance is

$$\int_s^{2s}\int_{y-2s+\frac{2s^2}{y}}^{s} dy\,dx = \int_s^{2s}\left(3s - y - \frac{2s^2}{y}\right)dy$$

$$= 3s^2 - \tfrac{3}{2}s^2 - 2s^2\log 2 = \tfrac{3}{2}s^2 - 2s^2\log 2,$$

and the denominator is the same as before; consequently, the required chance is $\frac{3}{4} - \log 2$.

So the chance that either of other segments subtends an obtuse angle is $\frac{3}{4} - \log 2$; and as the three cases in which an obtuse-angled triangle is made are mutually exclusive, the chance that an obtuse-angled triangle is made at all is $\frac{9}{4} - 3\log 2$; and the chance that a triangle is made being $\frac{1}{4}$, the chance that an acute-angled triangle is made is

$$\tfrac{1}{4} - (\tfrac{9}{4} - 3\log 2) = 3\log 2 - 2.$$

[For other solutions, see *Reprints*, II., 74, 75; V., 33; XV., 35, 36.]

6966. (By the Rev. W. A. Whitworth, M.A.)—Show that there are 220 triangles whose sides are integers not exceeding 100, and whose areas are also integral.

Solution by G. Heppel, M.A.; B. Reynolds, M.A.; *and others.*

It is impossible, without going to a great length, to do more than give a sketch of the method of determinining the triangles. It is easy to see that the sines and cosines of all the angles must be rational fractions. Using the general form $\frac{m^2-n^2}{m^2+n^2}$, $\frac{2mn}{m^2+n^2}$, make a list of rational sines and cosines with denominators not greater than 100,

rejecting all that are not in their lowest terms. There will be 16 of these, which will give us at once as many right-angled triangles. Of some of these multiples may be taken, for example, multiples up to 20 of 3, 4, 5; multiples up to 7 of 5, 12, 13, and so on. In this way there will be found to be 52 right-angled triangles.

Next, to every such right-angled triangle with sides abc, (c being hypotenuse) there must be two corresponding isosceles triangles $c, c, 2a$, and $c, c, 2b$. Of these there will be 23 with sides not greater than 100, and taking multiples as before, the number will be increased to 72.

Lastly, combine any other pairs of the original list of sines and cosines, determine a third fraction by the sin (A + B) formula, and take the least three integers proportional to the three fractions. Reject those containing numbers over 100, and those which are repetitious. This will furnish 56 scalene triangles, and with the multiples 96. Hence the total number is, as stated in the Question, 220.

The list of the 220 triangles is as set forth hereunder:—

(1) Right-angled; in number including multiples, 52.

3	4	5	20	21	29	11	60	61	36	77	85
5	12	13	12	35	37	16	63	65	13	84	85
8	15	17	9	40	41	33	56	65	39	80	89
7	24	25	28	45	53	48	55	73	65	72	97

(2) Isosceles; in number, including multiples, 72.

5	5	6	25	25	14	41	41	18	65	65	66
5	5	8	25	25	48	41	41	80	73	73	96
13	13	10	29	29	40	53	53	56	85	85	72
13	13	24	29	29	42	53	53	90	85	85	26
17	17	16	37	37	24	61	61	22	89	89	78
17	17	30	37	37	70	65	65	32			

(3) Scalene; in number, including multiples, 96.

4	13	15	25	29	36	34	65	93	53	75	88
13	14	15	25	39	40	35	53	66	57	82	89
7	15	20	29	35	48	36	61	65	56	61	75
11	13	20	39	41	50	37	91	96	57	65	68
10	17	21	13	68	75	39	41	50	60	73	91
12	17	25	15	41	52	39	85	92	61	74	87
13	20	21	17	55	60	40	51	77	61	91	100
17	25	26	20	37	51	41	51	58	65	76	87
17	25	28	25	39	56	41	84	85	65	87	88
13	37	40	25	52	63	43	61	68	68	75	77
13	40	45	25	51	52	48	85	91	68	87	95
15	34	35	25	74	77	50	69	73	75	86	97
15	37	44	26	51	55	51	52	53	78	95	97
17	39	44	29	52	69	52	73	75	89	99	100.

6872. (By R. Knowles, B.A., L.C.P.)—In a central conic, if GK is the perpendicular from G, the foot of the normal at P on CP drawn from the centre C, prove that the rectangle CP . PK is constant.

Solution by A. COHEN, Q.C., M.P.; SARAH MARKS; *and others.*

Draw CU perpendicular on the tangent. Then triangles CUP, GPK are similar. Therefore PK . PC = PG . CU. But the product of the normal and the perpendicular on the tangent equals b^2. Therefore PK . PC = b^2.

7001. (By T. WOODCOCK, B.A.)—The straight line through P, any point on the circumscribing circle of a triangle ABC, drawn perpendicular to the pedal line of P, meets the sides of the triangle in x, y, z: and the straight line through the orthocentre, parallel to the pedal line of P, meets the sides in X, Y, Z; Pp, Pq, Pr are drawn at right angles to PX, PY, PZ to meet the corresponding sides in p, q, r; show that (1) xp, yq, zr are bisected by the pedal line of P; (2) the triangle pqr is double of the triangle ABC; (3) PAq, PrA; PBr, PpB; and PCp, PqC are pairs of similar triangles.

Solution by the PROPOSER; E. RUTTER; *and others.*

The lines XYZ, xyz, and the pedal line of P will be the directrix, axis, and tangent at the vertex respectively, of a parabola with focus P and touching the sides of the triangle in p, q, r: and the above are well known properties of this parabola.

6809. (By J. L. MCKENZIE, B.A.)—(1) Let O be the orthocentre of a triangle ABC, and N the centre of the nine-points circle. Let the circle through O, and through the middle points of OB and OC, cut the line NO in P; and let the two other analogous circles cut NO in Q and R. Prove that the lines joining P, Q, and R with the middle points of BC, BA, and AB, all intersect in a point S on the circumference of the nine-points circle. (2) Let the circle through N, and through the middle points of OB and OC, cut NO in P′, and similarly for two other circles; prove that the lines joining P′, Q′, R′ to the feet of the perpendiculars on BC, CA, and AB from the opposite angles, all intersect in the same point S as before.

Solution by the PROPOSER; Professor MATZ, M.A.; *and others.*

It is easily seen, on drawing a figure, that each of the three circles in the first part of the question is equal to the nine-point circle; and each circle in (2) has a diameter equal to half the diameter of a circle that passes through two angular points of the triangle ABC, and through the centre of the circumscribing circle. It can be proved, that if T be the other extremity of the chord of the nine-points circle drawn through S, the points T, N, O, and the middle points of OA, OB, and OC, are six points on the same hyperbola.

6790. (By G. F. WALKER, M.A.)—Three particles are at the corners of an equilateral triangle of side a, and are each projected parallel to the opposite side towards the same parts with a velocity $[af(a)]^{\frac{1}{2}}$, where $f(r)$ is the law of attraction between the particles; show that they will always be at the corners of an equilateral triangle.

Solution by the PROPOSER; Professor SEITZ, M.A.; *and others.*

Suppose a smooth wire ring to be drawn through the particles and that they are constrained to slide on it. Let R be the pressure. The particles being all set off in the same direction round the circle, and with the same velocity, it is clear that their motion will continue unaltered. We have then $m\frac{af(a)}{a} = \mathrm{R} + \frac{1}{2} \cdot 2f(a) \cdot m$, therefore $\mathrm{R} = 0$; and the wire may be removed without altering the motion.

6989. (By R. TUCKER, M.A.)—Show that, in the curve whose equation is $b^2 e^{\frac{y}{b}} + a^2 e^{-\frac{y}{b}} = 2a^2 \cos\frac{x}{b}$, there are pairs of radii of curvature (ρ, ρ'), so related that $\rho^2 + \rho'^2 = a^2 + b^2$.

Solution by the PROPOSER.

Referring to Vol. I. of the *Reprint*, it will be seen that the radial of the curve in question is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, of which it is a well-known property that $\mathrm{CP}^2 + \mathrm{CD}^2 = a^2 + b^2$; whence, &c.

6895. (By Professor MATZ, M.A.)—Find the value of ϕ in the equation $\pi \tan^2\phi + \sqrt{(2)} - \sqrt{(3)} = -3\,[\pi\sqrt{(6)}\tan^2\phi]^{\frac{1}{3}}$.

Solution by W. H. BLYTHE, B.A.; H. L. ORCHARD, M.A.; *and others.*

The equation is of the form $(a+b+c)(a^2+b^2+c^2-ab-bc-ca) = 0$; where $a^3 = \pi\tan^2\phi$, $b^3 = \sqrt{2}$, $c^3 = -\sqrt{3}$. Since b and c are real and unequal, the roots of the quadratic factor are imaginary; taking, therefore, $a+b+c = 0$, and solving for $\tan\phi$, we find $42'\,38''{\cdot}5$ as the least value for the angle ϕ, all values included in $n\pi \pm \alpha$.

6944. (By W. Gallatly, B.A.)—Prove that, in any triangle ABC,

$$\frac{\sin\frac{1}{2}(B-C)}{\sin\frac{1}{2}(B+C)}+\frac{\sin\frac{1}{2}(C-A)}{\sin\frac{1}{2}(C+A)}+\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}$$
$$+\frac{\sin\frac{1}{2}(B-C)\,.\,\sin\frac{1}{2}(C-A)\,.\,\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(B+C)\,.\,\sin\frac{1}{2}(C+A)\,.\,\sin\frac{1}{2}(A+B)}=0.$$

Solution by A. McMurchy, B.A.; J. O'Regan; *and others.*

The first term is $\dfrac{\sin B-\sin C}{\sin A}=\dfrac{b-c}{a}$, and, treating all the terms similarly, we have $\dfrac{b-c}{a}+\dfrac{c-a}{b}+\dfrac{a-b}{c}+\dfrac{(b-c)(c-a)(a-b)}{abc}$,

which vanishes identically.

6756. (By E. B. Elliott, M.A.)—If u be a homogeneous function of the n^{th} degree in x_1, x_2, x_3, x_4, four variables u_1, u_2, u_3, u_4 its first, and $u_{11}, u_{12}, \ldots u_{44}$ its second partial differential coefficients, prove (1) that

$$\begin{vmatrix} u_{11}, \ldots, u_{14} \\ \ldots\ \ldots\ \ldots \\ u_{41}, \ldots, u_{44} \end{vmatrix} = \frac{(n-1)^2}{x_4^2}\begin{vmatrix} u_{11}, u_{12}, u_{13}, u_1 \\ u_{21}, u_{22}, u_{23}, u_2 \\ u_{31}, u_{32}, u_{33}, u_4 \\ u_1, u_2, u_3, 0 \end{vmatrix} + \frac{n(n-1)}{x_4^2}u\begin{vmatrix} u_{11}, u_{12}, u_{13} \\ u_{21}, u_{22}, u_{23} \\ u_{31}, u_{32}, u_{33} \end{vmatrix};$$

and (2) that corresponding relations hold for any number of variables.

Solution by W. J. C. Sharp, M.A.; Professor Nash, M.A.; *and others.*

The value of the Hessian given in the Question is an expansion of the expression

$$\frac{n-1}{x_4^2}\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & u_1 \\ u_{12}, & u_{22}, & u_{23}, & u_2 \\ u_{13}, & u_{23}, & u_{33}, & u_3 \\ (n-1)u_1, & (n-1)u_2, & (n-1)u_3, & nu \end{vmatrix},$$

which I gave in a note to a solution of Quest. 5762 (*Reprint*, Vol. xxxiii., p. 92). The same method gives the generalized result

$$\frac{(n-1)^2}{x_n^2}\begin{vmatrix} u_{11}, & u_{12}, & \ldots, & u_{1,n-1}, & u_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ u_{1,n-1}, & u_{2,n-1}, & \ldots, & u_{n-1,n-1}, & u_{n-1} \\ u_1, & u_2, & \ldots, & u_{n-1}, & 0 \end{vmatrix}$$
$$+\frac{n(n-1)}{x_n^2}\begin{vmatrix} u_{11}, & u_{12}, & \ldots, & u_{1,n-1} \\ u_{12}, & u_{22}, & \ldots, & u_{2,n-1} \\ \ldots & \ldots & \ldots & \ldots \\ u_{1,n-1}, & u_{2,n-1}, & \ldots, & u_{n-1,n-} \end{vmatrix}.$$

These forms furnish a direct proof that, no matter how many the variables, the Hessian is satisfied by all the singular roots (Salmon's *Higher Algebra*, p. 85) of the quantic.

6764. (By the Rev. T. R. Terry, M.A., F.R.A.S.)—Show that, unless m is a negative integer, the limit, when p is indefinitely increased, of

$$\frac{m\,(m+1)\,(m+2)\,\ldots\ldots\,(m+2p-1)}{1\,.\,3\,.\,5\,\ldots\,(2p-1)\,2m\,(2m+2)\,\ldots\,(2m+2p-2)} \text{ is } 2^{m-1}.$$

Solution by E. Haigh, B.Sc., B.A.; Belle Easton; *and others.*

Let $$u = \frac{m\,(m+1)\,(m+2)\,\ldots\ldots\,(m+2p-1)}{1\,.\,3\,.\,5\,\ldots\,(2p-1)\,2m\,(2m+2)\,\ldots\,(2m+2p-2)},$$

then $$u = \frac{m+2p-1\,!\;\;p\,!}{m+p-1\,!\;\;2p\,!} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

m being assumed to be a positive integer.

By a known theorem, when x is infinite, the limit of $1\,.\,2\,\ldots\,x$ is

$$(2\pi x)^{\frac{1}{2}}\,x^x\,\epsilon^{-x} \text{ or } Kx^{x+\frac{1}{2}}\epsilon^{-x},$$

where K is a constant. Hence, in the limit, when p is infinite,

$$u = \frac{(m+2p-1)^{m+2p-\frac{1}{2}}\;p^{p+\frac{1}{2}}}{(m+p-1)^{m+p-\frac{1}{2}}\,(2p)^{2p+\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

ϵ disappearing from the ratio by virtue of the relation

$$(m+2p-1)+p = (m+p-1)+2p.$$

Again, let x and z be two infinite quantities of the same order, whose difference is always finite and ratio is unity. Let $a = m-1$; then

$$(x+a)^z = x^z\left(1+\frac{a}{x}\right)^z = x^z\left\{1+z\,\frac{a}{x}+\frac{z\,(z-1)}{1\,.\,2}\,\frac{a^2}{x^2}+\ldots\right\}$$

$$= x^z\left\{1+\frac{z}{x}a+\frac{\frac{z}{x}\left(\frac{z}{x}-\frac{1}{x}\right)}{1\,.\,2}a^2+\ldots\right\} = x^z\left\{1+a+\frac{a^2}{1\,.\,2}+\ldots\right\} = x^z\epsilon^a.$$

Putting $x = 2p$, $z = m+2p-\frac{1}{2}$, we obtain

$$(m+2p-1)^{m+2p-\frac{1}{2}} = \epsilon^{m-1}\,(2p)^{m+2p-\frac{1}{2}};$$

and, similarly, by substituting $x = p$, $z = m+p-\frac{1}{2}$,

$$(m+p-1)^{m+p-\frac{1}{2}} = \epsilon^{m-1}\,p^{m+p-\frac{1}{2}}.$$

Hence, substituting in (2),

$$u = \frac{(2p)^{m+2p-\frac{1}{2}}\;p^{p+\frac{1}{2}}}{p^{m+p-\frac{1}{2}}\,(2p)^{2p+\frac{1}{2}}} = 2^{m-1} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

[Putting $a = \frac{1}{2}m$, $r = 2$ in the well known formula

$$\Gamma(a)\,\Gamma\left(a+\frac{1}{r}\right)\ldots\Gamma\left(a+\frac{r-1}{r}\right) = \Gamma(ra)\,.\,(2\pi)^{\frac{r-1}{2}}\,.\,r^{\frac{1}{2}-ra},$$

we have $$\frac{\Gamma(\frac{1}{2})\;\Gamma(m)}{\Gamma(\frac{1}{2}m)\;\Gamma[\frac{1}{2}(m+1)]} = 2^{m-1}.$$

But $\Gamma(x)$ is the limit of $$\frac{1\,.\,2\,.\,3\,\ldots\,p}{x\,(x+1)\,\ldots\,(x+p-1)}\,p^{x-1},$$

whence, substituting, we get the required result.]

6876. (By the Rev. W. A. WHITWORTH, M.A.)—Without altering the perimeter, it is required to lengthen very slightly one of the sides of an equilateral triangle and to shorten another, so that the radius of the inscribed circle may become commensurable with the sides. The increment and decrement, in order to be inappreciable to the eye, must not exceed $\frac{1}{100}$th part of the length of the sides. Determine the ratio in which the sides may be altered.

Solution by G. M. REEVES, M.A.; CHRISTINE LADD, B.A.; *and others.*

Let $a+x$, a, $a-x$ be the sides of the slightly altered triangle; then

$$r = \frac{a}{2\sqrt{3}}\left(1-\frac{4x^2}{a^2}\right)^{\frac{1}{2}};\quad \text{hence, putting } \frac{2x}{a} = \frac{1}{y},\ \text{we have}$$

$$\tfrac{1}{3}\left(1-\frac{1}{y^2}\right) = \text{perfect square} = s^2 \text{ (suppose)};\quad \text{whence } y^2-3v^2 = 1,$$

where $v = ys$; and the ordinary method of solution gives

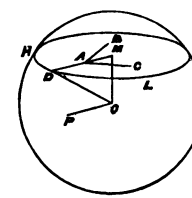
$$y = \tfrac{1}{2}\left[(2+\sqrt{3})^n + (2-\sqrt{3})^n\right];$$

and thence, for $n = 5$, we have $y = 362$; therefore $x = \frac{1}{724}a$, and the sides then become proportional to 723, 724, 725, and r becomes $\frac{362}{724}$ of the mean side.

6542 & 6628. (By Professor SEITZ, M.A.) — A plane is passed through three random points within a sphere; show (1) that the average area of the section is $\frac{10}{11}\pi r^2$, and (2) that the average volume of the smaller of the two segments into which the sphere is divided is $\frac{697}{1536}\pi r^3$.

Solution by Prof. CAVALLIN, M.A.; *the* PROPOSER; *and others.*

Let A, B, C be the three random points, DHKL the section made by the plane through A, B, C, M its centre, O the centre of the sphere, and OP a radius of the sphere, such that AB is parallel to the plane MOP.



Let OD $= r$, AM $= x$, AB $= y$, AC $= z$, MD $= x'$, u = area DHKL, v = the volume of smaller segment, $\angle$DOM $= \theta$,

$\angle$MAB $= \phi$, $\angle$CAM $= \downarrow$, $\angle$MOP $= \mu$,

and the angle the plane MOP makes with a fixed plane through OP $= \omega$.

Then $x' = r\sin\theta$, $u = \pi r^2\sin^2\theta$, $v = \frac{1}{3}\pi r^3(2-\sin^2\theta\cos\theta)$;

an element of the sphere at A is $r\sin\theta\, d\theta\,.\,2\pi x\, dx$, at B it is $y^2\, dy\, d\phi\, d\mu$, and at C it is $\sin(\phi+\downarrow)\sin\mu\, z^2\, dz\, d\downarrow\, d\omega$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of x, 0 and x', and tripled; of ϕ, $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; of $\downarrow$, $-\phi$ and $\frac{1}{2}\pi$, and doubled; of μ, 0 and π; of ω, 0 and 2π; of y, 0 and $2x\cos\phi = y'$; and of z, 0 and $2x\cos\phi = z'$. The whole number of ways the three points can be taken is $(\frac{4}{3}\pi r^3)^3$.

1. The average area of the section of the sphere is

$$\left(\tfrac{6}{\frac{4}{3}\pi r^3}\right)^3 \int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{-\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{y'}\int_0^{z'} ur\sin\theta\, d\theta \cdot 2\pi\, x\, dx\, d\phi \sin(\phi+\psi) \times \sin\mu\, d\mu\, d\omega\, y^2\, dy\, z^2\, dz$$

$$= \frac{36}{\pi^2 r^8}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi} u\sin\theta\sin(\phi+\psi)\, d\theta\, x^7\, dx\cos^3\phi\, d\phi\cos^3\psi\, d\psi \times \sin\mu\, d\mu\, d\omega$$

$$= \frac{144}{\pi r^8}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi} u\sin\theta\sin(\phi+\psi)\, d\theta\, x^7\, dx\cos^3\phi\, d\phi\cos^3\psi\, d\psi$$

$$= \frac{18}{\pi r^8}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\left[3\left(\tfrac{1}{2}\pi+\phi\right)\sin\phi + 2\cos\phi + \sin^2\phi\cos\phi\right] u\sin\theta\, d\theta\, x^7\, dx \times \cos^3\phi\, d\phi$$

$$= \frac{315}{16r^8}\int_0^{\frac{1}{2}\pi}\int_0^{x'} u\sin\theta\, d\theta\, x^7\, dx = \frac{315\pi r^2}{128}\int_0^{\frac{1}{2}\pi}\sin^{11}\theta\, d\theta = \tfrac{10}{11}\pi r^2.$$

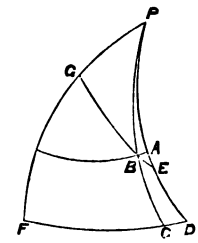
2. The average volume of the smaller segment is

$$= \frac{315}{16r^8}\int_0^{\frac{1}{2}\pi}\int_0^{x'} v\sin\theta\, d\theta\, x^7\, dx = \frac{105\pi r^3}{128}\int_0^{\frac{1}{2}\pi}(2-2\cos\theta-\sin^2\theta\cos\theta)\sin^9\theta\, d\theta = \frac{667}{1536}\pi r^3.$$

4862. (By Professor MARTIN, M.A.)—On a horizontal plane there is a hill in the form of a spherical segment, whose base is $2a$ rods in diameter, and its altitude or perpendicular height is b rods. A roadway winds round this hill, so as to be inclined at every point at angle β to the horizontal plane. Find (1) the *curve* the road is, (2) the whole length of the road, and (3) how high up the hill the road can ascend.

Solution by the PROPOSER.

If a line be drawn tangent to the hill at a point $r\cos\beta-(r-b)$ perpendicularly above the plane of its base, it will make an angle β with the plane of the base of the hill; therefore the road cannot ascend above this point, which is $r\beta$ from the top of the hill measured on its surface, r being the radius of the sphere of which the hill is a segment.



Let P be the summit of the hill, GBE a portion of the road (G being the upper extremity), FCD part of the circumference of the base of the hemisphere of which the hill is a part, (PF being a meridian through G), PBC and PAED two meridians infinitely near each other, and BA a small portion of the circumference of a section of the hill through B parallel to its base.

Put θ = FC, $r\phi$ = PB, λ = GE, $r\delta$ = a meridian through the summit and circumference of the base of the hill; then

$$d\theta = \text{CD}, \quad r d\phi = \text{AE}, \quad d\lambda = \text{BE};$$

and $$\text{BA} : \text{CD} = \sin \text{PB} : \sin \text{PC};$$

or $$\text{BA} : d\theta = \sin\phi : 1; \text{ whence BA} = \sin\phi\, d\theta.$$

The triangle BAE, being infinitely small, may be considered a plane triangle right-angled at A; therefore

$$(\text{AE})^2 + (\text{AB})^2 = (\text{BE})^2, \text{ or } d\lambda^2 = r^2 d\phi^2 + \sin^2\phi\, d\theta^2 \quad \ldots\ldots\ldots(1).$$

Raise a perpendicular from E; draw a horizontal line through B to intersect this perpendicular, and a right-angled triangle will be formed whose perpendicular is the differential of the perpendicular from the vertex of the hill on a plane through B parallel with the base of the hill, and whose base makes an angle β with AE. Therefore

$$r \sin\phi\, d\phi = \text{AE} \sin\beta, \text{ and } \text{AE} = d\lambda = \frac{r \sin\phi\, d\phi}{\sin\beta} \quad \ldots\ldots\ldots (2).$$

Equating the two values of $d\lambda$, we get

$$d\theta = \frac{r d\phi\, (\sin^2\phi - \sin^2\beta)^{\frac{1}{2}}}{\sin\beta \sin\phi} = \frac{r\,(\sin^2\phi - \sin^2\beta)\, d\phi}{\sin\beta \sin\phi\, (\sin^2\phi - \sin^2\beta)^{\frac{1}{2}}},$$

$$= \frac{r \sin\phi\, d\phi}{\sin\beta\, (\sin^2\phi - \sin^2\beta)^{\frac{1}{2}}} - \frac{r \sin\beta\, d\phi}{\sin\phi\, (\sin^2\phi - \sin^2\beta)^{\frac{1}{2}}}.$$

Integrating, we obtain

$$\theta = r \operatorname{cosec}\beta \cos^{-1}(\sec\beta \cos\phi) - r \sin\beta \cos^{-1}(\tan\beta \cot\phi) + \text{C}.$$

When $\theta = 0$, $\phi = \beta$; therefore C = 0, and

$$\theta = r \operatorname{cosec}\beta \cos^{-1}(\sec\beta \cos\phi) - r \sin\beta \cos^{-1}(\tan\beta \cot\phi)$$

is the equation to the road, where $r = \frac{a^2 + b^2}{2b}$.

Integrating (2), $$\lambda = -\frac{r \cos\phi}{\sin\beta} + \text{C}.$$

When $\lambda = 0$, $\phi = \beta$; therefore $\text{C} = \frac{r \cos\beta}{\sin\beta}$, and $\lambda = \frac{r(\cos\beta - \cos\phi)}{\sin\beta}$.

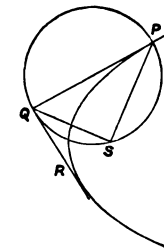
When $\phi = \delta$, $\lambda = \frac{r\,(\cos\beta - \cos\delta)}{\sin\beta} = r \cot\beta - (r - b) \operatorname{cosec}\beta$,

the length of the road.

When the hill is a hemisphere, $b = r$ and $\lambda = r \cot\beta$

6976. (By J. Young, B.A.)—Draw a common tangent to a parabola and any circle intersecting it and passing through the focus of the parabola.

Solution by the PROPOSER; J. O'REGAN; *and others.*



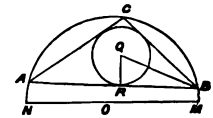
At a point P where the circle cuts the parabola, draw a tangent to the parabola; at the point Q where this tangent again meets the circle, draw a tangent to the circle. This line will be a common tangent, for, if S be the focus, $\angle$ SQR = SPQ from the tangent property of the circle; and this is the condition that QR should be a tangent to the parabola.

As the circle cuts the parabola in two points, these are two real common tangents.

6600. (By Professor SEITZ, M.A.)—A plane triangle is formed by joining three points taken at random in a given semi-circumference; show that the average area (1) of the triangle is $\frac{3r^2}{\pi}\left(1-\frac{8}{\pi^2}\right)$, and (2) of its inscribed circle is $\pi r^2\left(\frac{5}{2}-\frac{162}{\pi^2}+\frac{432}{\pi^3}\right)$.

Solution by the PROPOSER.

Let MCN be the given semi-circumference, O its centre, A, B, C the three random points, and Q the centre of the inscribed circle of ABC.



Let OM = r, QR = x, arc AM = $r\theta$, arc AB = $r\phi$, arc AC = $r\psi$, and area ABC = u.

Then AB = $2r\sin\frac{1}{2}\phi$, AC = $2r\sin\frac{1}{2}\psi$,

BR = $4r\cos\frac{1}{2}\phi\cos\frac{1}{2}\psi\sin\frac{1}{2}(\phi-\psi)$, $u = 2r^2\sin\frac{1}{2}\phi\sin\frac{1}{2}\psi\sin\frac{1}{2}(\phi-\psi)$,

and $x = \text{BR}\tan\frac{1}{2}\psi = 4r\cos\frac{1}{2}\phi\sin\frac{1}{2}\psi\sin\frac{1}{2}(\phi-\psi)$.

The limits of θ are 0 and π; of ϕ, 0 and θ; and of ψ, 0 and ϕ.

1. The average area of the triangle is

$$\frac{\int_0^\pi\int_0^\theta\int_0^\phi u\,r\,d\theta\,r\,d\phi\,r\,d\psi}{\int_0^\pi\int_0^\theta\int_0^\phi r\,d\theta\,r\,d\phi\,r\,d\psi} = \frac{6}{\pi^3}\int_0^\pi\int_0^\theta\int_0^\phi u\,d\theta\,d\phi\,d\psi$$

$$= \frac{6r^2}{\pi^3}\int_0^\pi\int_0^\theta(1-\cos\phi-\tfrac{1}{2}\phi\sin\phi)\,d\theta\,d\phi = \frac{3r^2}{\pi^3}\int_0^\pi(2\theta+\theta\cos\theta-3\sin\theta)\,d\theta$$

$$= \frac{3r^2}{\pi}\left(1-\frac{8}{\pi^2}\right).$$

2. The average area of the inscribed circle is

$$\frac{6}{\pi^3}\int_0^{\pi}\int_0^{\theta}\int_0^{\phi}\pi x^2\,d\theta\,d\phi\,d\psi = \frac{12r^2}{\pi^2}\int_0^{\pi}\int_0^{\theta}(2\phi+\phi\cos\tfrac{1}{2}\phi-6\sin\tfrac{1}{2}\phi)\cos^2\tfrac{1}{4}\phi\,d\theta\,d\phi$$

$$= \frac{3r^2}{2\pi^2}\int_0^{\pi}(5\theta^2+24\theta\sin\tfrac{1}{2}\theta+2\theta\sin\theta-110+96\cos\tfrac{1}{2}\theta+14\cos\theta)\,d\theta$$

$$= \pi r^2\left(\frac{5}{2}-\frac{162}{\pi^2}+\frac{432}{\pi^3}\right).$$

3083 & 5143. (By Dr. James Matteson.)—Find a common value of x that will make $x^2\pm ax = \square$, $x^2\pm bx = \square$, $x^2\pm cx = \square$, &c., *ad inf.*; and again $a^2x^2\pm ax = \square$, $b^2x^2\pm bx = \square$, $c^2x^2\pm cx = \square$, &c., *ad inf.*

Solution by Dr. D. S. Hart.

(3083). In order to solve this problem, it will be necessary to premise the following theorems:—(1) Every prime number of the form of $4t+1$ is the sum of two squares in one way only; (2) the square of every such number is also the sum of two squares in one way only; (3) the product of any two such numbers is the sum of two squares in two ways; the product of any three such numbers is the sum of two squares in four ways; the product of any four such numbers is the sum of two squares in eight ways; and, in general, the product of any (n) such numbers is the sum of two squares in 2^{n-1} ways. By attending to these theorems, the following solution will be readily understood.

Let $x = z^2$, then, by substitution and expunging z^2, we have $z^2\pm a = \square$, $z^2\pm b = \square$, $z^2\pm c = \square$, &c. *ad infinitum.*

Let $z^2 = m^2+n^2$, and $a = 2mn$, then $z^2\pm a = m^2\pm 2mn+n^2 = (m\pm n)^2$, also, let $z^2 = p^2+q^2$, and $b = 2pq$, then $z^2\pm b = p^2\pm 2pq+q^2 = (p\pm q)^2$; again, let $z^2 = r^2+s^2$, and $c = 2rs$, then $z^2\pm c = r^2\pm 2rs+s^2 = (r\pm s)^2$, and so on, *ad adfinitum.* Here all the conditions of the problem will be satisfied, if we can divide z^2 into two squares in any (n) number of ways. In order to do this, it will be necessary to solve the following problem: "To divide a given square number into two other square numbers." Let $A^2 = z^2 =$ the given square, and v^2, w^2 be the required squares; then $A^2 = v^2+w^2$, and $A^2 - w^2 = v^2$. Assume $A+w = \frac{p'v}{q'}$, and $A-w = \frac{q'v}{p'}$. Hence, we readily find

$$2A = \frac{p'v}{q'}+\frac{q'v}{p'} = \frac{(p'^2+q'^2)v}{p'q'}, \text{ and } 2w = \frac{p'v}{q'}-\frac{q'v}{p'} = \frac{(p'^2-q'^2)v}{p'q'};$$

whence $$v = \frac{2p'q'A}{p'^2+q'^2}, \text{ and } w = \frac{(p'^2-q'^2)A}{p'^2+q'^2};$$

in which formulæ A, or its equal z, may be taken any prime number or product of two or more prime numbers, each of which is the sum of two squares; then $p'^2+q'^2$ being put successively equal to each sum of two squares into which A or z can be divided, we shall have as many integral values of v, w; whence also $z^2 = v^2+w^2$ the same number of different ways, for which may be put successively m^2+n^2, p^2+q^2, r^2+s^2, &c. The different values of $2vw$ may be put successively

$$= 2mn,\ 2pq,\ 2rs,\ \&c. = a,\ b,\ c,\ \&c.$$

Let $A = z = 5 = 2^2+1^2$, and put $p' = 2$, $q' = 1$, then $v = 4$, $w = 3$, and $z^2 = 5^2 = 4^2+3^2$, being the sum of two squares in *one* way only. Let $A = z = 5 . 13 = 65$, then

$$5 = 2^2+1^2,\ 13 = 3^2+2^2,\ 65 = 8^2+1^2 = 7^2+4^2,$$

and, putting p', q' successively = the roots of each set of two squares, we shall have $v = 52$, $w = 39$; $v = 60$, $w = 25$; $v = 16$, $w = 63$; $v = 56$, $w = 33$; whence z^2 = sum of two squares in *four* ways. Let

$$z = 5 . 13 . 17 = 1105,$$

then
$$5 = 2^2+1^2,\ 13 = 3^2+2^2,\ 17 = 4^2+1^2;$$
$$5 . 13 = 65 = 8^2+1^2 = 7^2+4^2;\ 5 . 17 = 85 = 9^2+2^2 = 7^2+6^2;$$
$$13 . 17 = 221 = 11^2+10^2 = 14^2+5^2;$$
and
$$1105 = 33^2+4^2 = 32^2+9^2 = 31^2+12^2 = 24^2+23^2;$$

and, putting p', q' successively = the roots of each set of two squares, we shall have the following values of v, w, viz.: 884, 663; 1020, 425; 520, 975; 272, 1071; 952, 561; 468, 1001; 1092, 169; 1100, 105; 700, 855; 264, 1073; 576, 943; 744, 817; 1104, 47; whence z^2 = sum of two squares in *thirteen* ways.

In like manner, if z = the product of four factors, each of which is the sum of two squares; then, taking the factors singly, the product of every two, the product of every three, and finally, the product of all four, we shall find *forty* ways in which z^2 will be divided into the sum of two squares. And, if z = the product of five such factors, we shall, by proceeding as above, find *one hundred and twenty-one* ways in which z^2 will be divided into the sum of two squares, and so on for 6, 7, 8... to n factors. Whence we get the series 1, 4, 13, 40, 121, 364, 1093, &c., the law of continuation of which is $3m'+1$, m' being any term in the series. Since $1 = 1$, $4 = 3+1$, $13 = 3^2+3+1$, $40 = 3^3+3^2+3+1$, &c., it is evident that the n^{th} term of the series is the sum of a geometric series, whose first term is 1, ratio 3, and last term 3^{n-1}; therefore S = sum = $\frac{1}{2}(3^n-1)$, where n is the number of factors used in the n^{th} term.

Having thus found the values of z^2, a, b, c, &c. in the expressions $z^2 \pm a = \square$, $z^2 \pm b = \square$, $z^2 \pm c = \square$, &c., if these expressions be multiplied by $x = z^2$, we shall solve the original expressions $x^2 \pm ax = \square$, $x^2 \pm bx = \square$, $x^2 \pm cx = \square$, &c.

(5143). This problem can be solved in the same manner as Quest. 3083; for, if the expressions be divided by a^2, b^2, c^2, &c. respectively, we shall have

$$x^2 \pm \frac{1}{a}x = \square,\ x^2 \pm \frac{1}{b}x = \square,\ x^2 \pm \frac{1}{c}x = \square,\ \&c.\ \textit{ad infinitum};$$

and, putting $a' = \frac{1}{a}$, $b' = \frac{1}{b}$, $c' = \frac{1}{c}$, &c.,

the expressions become

$$x^2 \pm a'x = \square,\ x^2 \pm b'x = \square,\ x^2 \pm c'x = \square,\ \&c.\ ad.\ inf.,$$

which are of the same form as those in 3083, and may be solved in the same way; then, a', b', c', &c. being known, we have

$$a = \frac{1}{a'},\ b = \frac{1}{b'},\ c = \frac{1}{c'},\ \&c.$$

4722. (By Professor CAYLEY, F.R.S.)—1. Show that the conditions in regard to the reality of the roots of the equations $(x^2-a)^2 + 16A(x-m) = 0$, are, if $(4m^2-3a)^3 - (8m^3-9ma-27A)^2 = -$,

then the roots are two real, two imaginary; but if

$$(4m^2-3a)^3 - (8m^3-9ma-27A)^2 = +,$$

then, if simultaneously $a = +$, $A(ma-9A) = +$,

the roots are all real, but otherwise they are all imaginary.

2. If the roots of the foregoing equation are all imaginary, then for any real value whatever of y, the roots of the equation

$$(x^2+y^2-a)^2 + 16A(x-m) = 0$$

are all imaginary.

Solution by W. J. C. SHARP, M.A.

The invariants of $(x^2-a)^2 + 16A(x-m) = 0$,

or $x^4 - 2ax^2 + 16Ax + a^2 - 16Am$,

are $S = \frac{1}{3}(a^2 - 12Am)$ and $T = -\frac{5}{27}[a^3 - 18Ama + 54A^2]$,

and the discriminant

$$S^3 - 27T^2 = \tfrac{2^5}{3^7}A^2[(4m^2-3a)^3 - (8m^2-9ma+27A)^2],$$

and therefore (SALMON'S *Higher Algebra*, p. 175) the roots are two real and two imaginary if the quantity in the bracket be negative; and all real if that quantity be positive, and

$$(b^2-ac) \equiv \tfrac{1}{3}a \text{ and } 3aT + 2(b^2-ac)S \equiv \tfrac{1}{3^6}A(ma-9A)$$

be positive; and all imaginary if the quantity in the bracket be positive, and either $\frac{1}{3}a$ or $A(ma-9A)$ negative; and if $\frac{1}{3}a$ be negative, the quantity will continue so when $a-y^2$ is written for a, which proves (2).

The form $(x^2+y^2-a)^2 + 16A(x-m) = 0$ is that to which Professor CAYLEY has reduced the equation to a Cartesian; and hence it appears that those values of y which make $(4m^2+3y^2-a)^3 - (8m^2-9am+9my^2+27A)^2$ negative, correspond to two real and two imaginary values of x, whilst those which make the same quantity positive, give four real values of x if $a-y^2$ and $A(ma-my^2-9A)$ be both positive, and otherwise none.

7249. (By Professor SYLVESTER, F.R.S.)—If

$$1-\frac{1}{1-c}y+\frac{c}{(1-c)(1-c^2)}y^2+\ldots\pm\frac{c^{\frac{1}{2}(n^2-n)}}{(1-c)(1-c^2)\ldots(1-c^n)}y^n+\ldots=0,$$

prove that (1) the sum of the i^{th} powers of the several values of $\frac{1}{y}$ is $\frac{1}{1-c^i}$; and (2) the sum of the i^{th} homogeneous products of the same values is $\frac{1}{(1-c)(1-c^2)\ldots(1-c^i)}$.

[If $i=2$, the first sum is $\left(\frac{1}{1-c}\right)^2-2\frac{c}{(1-c)(1-c^2)}$, which is $\frac{1}{1-c^2}$, and the second sum is $\left(\frac{1}{1-c}\right)^2-\frac{1}{(1-c)(1-c^2)}$, which is $\frac{1}{(1-c)(1-c^2)}$.]

Solution by the Rev. T. R. TERRY, M.A.; G. HEPPEL, M.A.; *and others.*

By Algebra it may be shown that, if

$$A_n=\frac{c^n-1}{1-c^n}A_{n-1},\text{ and }B_n=\frac{1}{1-c^n}B_{n-1},$$

$$(1-y)(1-cy)(1-c^2y)\ldots=1-A_1y+A_2y^2-A_3y^3\ldots\quad\ldots\ldots(1),$$

and

$$\frac{1}{(1-y)(1-cy)(1-c^2y)\ldots}=1+B_1y+B_2y^2+\ldots\quad\ldots\ldots(2).$$

From (1) it follows that the given equation may be put in the form

$$(1-y)(1-cy)(1-c^2y)\ldots=0;$$

hence sum of n^{th} powers is $1+c^n+c^{2n}+\ldots=\frac{1}{1-c^n}$; and the sum of the n^{th} homogeneous products $=B_n=\frac{1}{(1-c)(1-c^2)\ldots(1-c^n)}$.

[If the equation stop with the term involving y^n, the values of the series of the powers and of the sums of the homogeneous products will be unaltered up to n dimensions.]

6856. (By Professor TOWNSEND, M.A.)—In the irrotational strain of an incompressible lamina in the plane of its mass, if the equations $r^n\cos n\theta=a^n$ and $r^n\sin n\theta=b^n$, where a and b are parameters, represent the line-systems of greatest and least displacement of the molecules; show that the equations $r^{\frac{1}{n}}\cos\frac{1}{n}n\theta=a^{\frac{1}{n}}$ and $r^{\frac{1}{n}}\sin\frac{1}{n}n\theta=b^{\frac{1}{n}}$, where a and b are parameters, represent the line-systems of greatest and least dilatation of the strain.

Solution by the PROPOSER.

Since generally, when in rectangular coordinates x and y, the equations $u=a$ and $v=b$, where u and v are conjugate functions of x and y, and where a and b are arbitrary parameters, represent in finite terms the line

systems of greatest or least molecular displacement in a strain such as that of the question, the differential equations

$$\frac{d^2u}{dx\,dy}\cdot\frac{dy}{dx}+\frac{d^2u}{dx^2}\pm\left[\left(\frac{d^2u}{dx\,dy}\right)^2+\left(\frac{d^2u}{dx^2}\right)^2\right]^{\frac{1}{2}}=0,$$

or their equivalents

$$\frac{d^2v}{dy^2}\cdot\frac{dy}{dx}+\frac{d^2v}{dx\,dy}\pm\left[\left(\frac{d^2v}{dy^2}\right)^2+\left(\frac{d^2v}{dx\,dy}\right)^2\right]^{\frac{1}{2}}=0,$$

represent, as is well known, the line systems of greatest and least dilatation of the strain; hence, when in polar coordinates $u=k\,.\,r^n\cos n\theta$ and $v=k\,.\,r^n\sin n\theta$, where k is a constant of the order of magnitude of the displacements determining the strain, since then

$$\frac{d^2u}{dx^2}=-\frac{d^2u}{dy^2}=\frac{d^2v}{dx\,dy}=+n(n-1)\,.\,k\,.\,r^{n-2}\cos(n-2)\theta,$$

$$\frac{d^2v}{dy^2}=-\frac{d^2v}{dx^2}=\frac{d^2u}{dx\,dy}=-n(n-1)\,.\,k\,.\,r^{n-2}\sin(n-2)\theta,$$

the aforesaid differential equations, after substitution for dy and dx of their equivalents $\sin\theta\,dr+r\cos\theta\,d\theta$ and $\cos\theta\,dr-r\sin\theta\,d\theta$, become in polar coordinates

$$[\cos(n-1)\theta\pm\cos\theta]\,dr-[\sin(n-1)\theta\pm\sin\theta]\,r\,d\theta=0,$$

or, for the upper and lower signs taken separately,

$$\frac{dr}{r}-\frac{\sin\frac{1}{2}n\theta\,d\theta}{\cos\frac{1}{2}n\theta}=0\quad\text{and}\quad\frac{dr}{r}+\frac{\cos\frac{1}{2}n\theta\,d\theta}{\sin\frac{1}{2}n\theta}=0,$$

and as the complete integrals of the latter infinite terms are respectively.

$$r^{\frac{1}{n}}\cos\tfrac{1}{2}n\theta=a^{\frac{1}{n}}\quad\text{and}\quad r^{\frac{1}{n}}\sin\tfrac{1}{2}n\theta=b^{\frac{1}{n}},$$

where a and b are the two constants of integration, therefore, &c.

Since evidently, from the aforesaid values of u and v,

$$\left[\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2\right]^{\frac{1}{2}}=\left[\left(\frac{dv}{dx}\right)^2+\left(\frac{dv}{dy}\right)^2\right]^{\frac{1}{2}}=nr^{n-1},$$

$$\left[\left(\frac{d^2u}{dx\,dy}\right)^2-\frac{d^2u}{dx^2}\cdot\frac{d^2u}{dy^2}\right]^{\frac{1}{2}}=\left[\left(\frac{d^2v}{dx\,dy}\right)^2-\frac{d^2v}{dx^2}\cdot\frac{d^2v}{dy^2}\right]^{\frac{1}{2}}=n(n-1)\,r^{n-2},$$

therefore, in the strain forming the subject of the question, the molecular displacement and principal dilatation depend throughout the mass only on the distance from the centre of the strain, and vary respectively as the $(n-1)^{\text{th}}$ and as the $(n-2)^{\text{th}}$ powers of that distance.

7255. (By Professor MARTIN, M.A., Ph.D.)—An urn contains 12 balls of which either 5, or 6, or 7, or 8, are white (but which one of these numbers is not known), and the rest are black; if a ball is drawn at random, find the chance that it is white.

Solution by the Rev. T. R. Terry, M.A.; E. Rutter; *and others.*

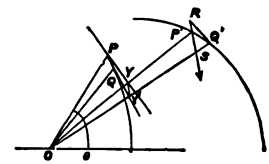
We assume that it is equally likely that there will be 5 or 6, or 7 or 8, white balls in the urn, and the chance required is obviously

$$= \frac{1}{4}\left(\frac{5+6+7+8}{12}\right) = \frac{13}{24}.$$

7259. (By Professor Minchin, M.A.) — A point P describes an orbit, with acceleration f, directed always to a fixed centre O; prove that the acceleration of the corresponding point P′, of the Hodograph, is in general, non-central, and makes with the radius vector OP′ an angle χ given by the equation $\tan\chi = \dfrac{\frac{d}{d\theta}\log\frac{f}{r}}{1-\cot\phi\,\frac{d}{d\theta}\log f}$, where ϕ is the angle between OP and the tangent at P, r = OP, and θ is the polar angle of r.

Solution by J. J. Walker, M.A.

Let PQ be an element of the path, P′Q′ the corresponding element of the hodograph, regarded as the inverse of the pedal, OY, OY′ being perpendiculars on the tangents at P, Q. Q′P′ representing the velocity at P′, let Q′R represent that at Q′, then RP′ is the direction of the acceleration at P′, and



$$\mathrm{OP'S} = \chi = (\mathrm{OP'Q'} \text{ or } \mathrm{OY'Y} \text{ or}) \ \phi - \mathrm{Q'P'S};$$

hence $$\tan\chi = \frac{\cot \mathrm{Q'P'S} - \cot\phi}{\cot\phi\cot \mathrm{Q'P'S} + 1}.$$

But $\cot \mathrm{Q'P'S} = \dfrac{\mathrm{Q'R} - \mathrm{Q'P'}}{\mathrm{Q'R}\,d\theta} = -\dfrac{d\log f}{d\theta}$; and $\cot\phi = -\dfrac{dr}{r\,d\theta} = -\dfrac{d\log r}{d\theta}$.

Thus $$\tan\chi = -\frac{\frac{d\log\frac{f}{r}}{d\theta}}{1-\cot\phi\,\frac{d\log f}{d\theta}},$$

the sign depending on the convention in taking χ.

The acceleration (a) in the hodograph can be deduced thus:—

$$\mathrm{RP'} = a\,dt = \mathrm{Q'R}\,\frac{d\theta}{\sin \mathrm{Q'P'S}}, \text{ or } a \propto \frac{f\,d\theta}{\sin \mathrm{P'}\,dt} \propto \frac{f}{r^2}\left\{1+\left(\frac{d\log f}{d\theta}\right)^2\right\}^{\frac{1}{2}},$$

where r = OP.

7060. (By R. Tucker, M.A.)—Three parabolas are described about the triangle ABC, so that A is the vertex of the diameter bisecting BC, and so on; show that (1) if p, q, r are the respective parameters of the diameters, then $\frac{1}{4}[p^{-2}+q^{-2}+r^{-2}] = 2\Sigma a^2 . \Sigma a^{-4} - 3\Sigma a^{-2}$.....($\alpha$),

$$a^4p^{-2}+\ldots = 12(a^2+b^2+c^2), \quad a^8p^{-4}+\ldots = 144(a^4+b^4+c^4).....(\beta, \gamma);$$

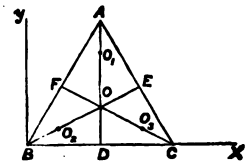
(2) when ABC is an equilateral triangle, the area of the triangle formed by joining the foci $= \frac{49}{64}\Delta ABC$.

Solution by R. Knowles, B.A.; J. Hammond, M.A.; *and others.*

1. Here $p . AD = DC^2$, therefore

$$p^2 = \frac{a^4}{8(b^2+c^2-\frac{1}{2}a^2)},$$

and similarly for q^2 and r^2. Therefore



$$\tfrac{1}{4}(p^{-2}+q^{-2}+r^{-2})$$
$$= \frac{2}{a^4b^4c^4}[b^4c^4(b^2+c^2-\tfrac{1}{2}a^2)+a^4c^4(a^2+c^2-\tfrac{1}{2}b^2) + a^4b^4(a^2+b^2-\tfrac{1}{2}c^2)],$$

which is identically the same as

$$2(a^2+b^2+c^2)\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right) - 3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right),$$

which proves (α); and (β) and (γ) may be proved similarly.

2. The triangle ABC being equilateral, the lines AD, &c. are the axes of the parabolas and A, B, C the vertices. Therefore

$$4O_1A \times AD = DC^2, \text{ therefore } O_1A = O_2B = O_3C = \tfrac{1}{24}\sqrt{3}\,a,$$

and the coordinates of O_1, O_2, O_3 are respectively

$$(\tfrac{1}{4}a, \tfrac{11}{24}\sqrt{3}\,a)\ (\tfrac{1}{16}a, \tfrac{1}{48}\sqrt{3}\,a)\ (\tfrac{13}{16}a, \tfrac{1}{48}\sqrt{3}\,a),$$

and $2\Delta O_1O_2O_3 = \frac{11}{24}\sqrt{3}a^2(\frac{1}{16}-\frac{13}{16}) + \frac{1}{48}\sqrt{3}a^2(\frac{13}{16}-\frac{1}{4}) + \frac{1}{48}\sqrt{3}a^2(\frac{1}{4}-\frac{1}{16})$

$$= \tfrac{7}{192}\sqrt{3}(\tfrac{1}{4}+\tfrac{1}{4}-11) = \tfrac{49}{64}\cdot\tfrac{1}{4}\sqrt{3}\,a^2,$$

therefore $\Delta O_1O_2O_3 = \frac{49}{64}\Delta ABC$.

[Otherwise: $OA = \frac{2}{3}AD = \frac{1}{3}a\sqrt{3} = \frac{8}{24}a\sqrt{3}$; therefore

$$OO_1 = \tfrac{8}{24}a\sqrt{3} - \tfrac{1}{24}a\sqrt{3} = \tfrac{7}{24}a\sqrt{3}, \text{ therefore } \frac{OO_1}{OA} = \frac{7}{8};$$

and the triangles $O_1O_2O_3$, ABC are similar, therefore $\Delta O_1O_2O_3 = \frac{49}{64}\Delta ABC$.]

6221. (By Professor Wolstenholme, M.A.)—Trace the curve

$$\frac{12a}{\pi}\int_0^\infty \frac{\sin z \sin 2z \sin 3z \sin 4z}{z^4}\cos\frac{xz}{a}\,dz,$$

and prove that its area (between the curve and the axis of x) is $288a^2$.

Solution by the PROPOSER.

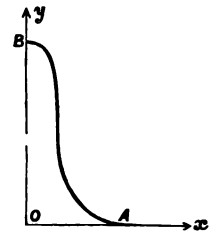
The figure is as annexed; and the values of $\frac{y}{a}$ are

$$\tfrac{1}{8}\left\{\left(8-\frac{x}{a}\right)^2\left(4+\frac{x}{a}\right)+16\right\}, \text{ if } \frac{x}{a}>0<6;$$

$$\tfrac{1}{4}\left\{3\left(9-\frac{x}{a}\right)^2+1\right\}, \text{ if } \frac{x}{a}>6<8;$$

$$\tfrac{1}{8}\left(10-\frac{x}{a}\right)^3, \text{ if } \frac{x}{a}>8<10;$$

and $\quad 0, \text{ if } \frac{x}{a}>10.$



The coordinates of the centroid of the area on the positive side of the axis of x are $\quad \bar{x} = \frac{1587}{720}a = 3{\cdot}607a, \quad y = \frac{17287}{1440}a = 12{\cdot}45a.$

Also for any negative value of x, y has the same value as for the same positive value. The shape of one half the curve is as in the figure. OA = $10a$, OB = $34a$ (rather too small in figure).

7201. (By R. F. SCOTT, M.A.)—Prove that

$$\int_0^{\pi} x \log\left(1-\sin^2 a \sin^2 x\right) dx = 2\pi^2 \log\cos\tfrac{1}{2}a.$$

Solution by R. KNOWLES, B.A.; C. MORGAN, M.A.; *and others.*

Given integral $\mathrm{I} = \int_0^{\frac{1}{2}\pi} \log\left(1-\sin^2 a \sin^2 x\right) dx = \pi \mathrm{J}$ say;

then
$$\frac{d\mathrm{J}}{da} = -2\sin a\cos a \int_0^{\frac{1}{2}\pi} \frac{\sin^2 x}{1-\sin^2 a\sin^2 x}\, dx$$

$$= -2\sin a\cos a\left(\tfrac{1}{2}\cdot\tfrac{1}{2}\pi + \sin^2 a\,\frac{1\cdot 3}{2\cdot 4}\tfrac{1}{2}\pi + \ldots\right)$$

$$= -\pi\frac{\cos a}{\sin a}\left(\frac{1}{\cos a}-1\right) = -\pi\tan\tfrac{1}{2}a,$$

therefore $\quad \mathrm{J} = 2\pi\log\cos\tfrac{1}{2}a + \mathrm{C}.$

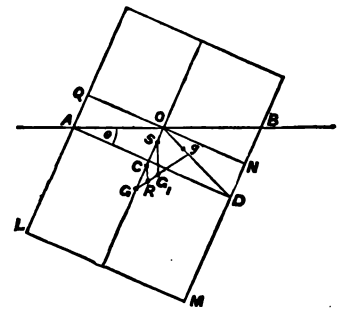
To find C, put $a = \frac{1}{2}\pi$, therefore $2\log\cos x = \pi\log\frac{1}{2} + \mathrm{C}$, therefore $\mathrm{C} = 0$, therefore $\mathrm{I} = 2\pi^2\log\cos\frac{1}{2}a$.

6933. (By J. J. WALKER, M.A.) — A homogeneous rectangular parallelepiped, the edges of which are a, b, c, floats in a liquid (the ratio of the densities of the two being $\rho : 1$) with the edge a vertical; show that,

if displaced through an angle θ (the top remaining above the surface of the liquid), so that the plane of ab remains parallel to itself, the body will right itself if a^2+b^2 is less than $(1+\sec^3\theta)+12\rho(1-\rho)$; and hence find the condition of stability for small displacements.

Solution by D. EDWARDES; Professor NASH, M.A.; *and others.*

Let the plane of floatation in the inclined position of the solid cut the vertical plane through the centroid in the line AB. Let C, G_1 be the centroids of the body and of the immersed portions respectively, G that of the rectangle ALMD, and g that of the triangle ADB. Let MN $(=h)$ be the length of axis which is immersed when the solid is vertical. Let O be the middle point of AB, and let CR, G_1S be drawn vertically. The solid will right itself if $SG_1 > CR$. Now



$$GC = \tfrac{1}{2}a - \tfrac{1}{2}h + \tfrac{1}{2}b\tan\theta,$$

$OC = h - \tfrac{1}{2}a$, also $\angle GCR = \theta$, whence $CR = \dfrac{b(a-h+\frac{1}{2}b\tan\theta)}{6h\sin\theta + b(1+\cos^3\theta)\sec\theta}$.

Again, $$GG : G_1g = b\tan\theta : 2h - b\tan\theta,$$

whence $$G_1S = \frac{b}{2h}Og = \frac{b^2}{12h}\sec\theta.$$

$$\therefore \quad b(a-h+\tfrac{1}{2}b\tan\theta)\cos\theta > \frac{b^2}{12h}[6h\sin\theta + b(1+\cos^2\theta)\sec\theta],$$

or $$12h(a-h)\cos^3\theta < b^2(1+\cos^2\theta),$$

or, since $h = a\rho$, $$12a\rho(1-\rho) < b^2(1+\sec^2\theta).$$

When θ is small, this condition becomes $6a^2\rho(1-\rho) < b^2$; and, if Ak^2 be the moment of inertia of the section bc about a line through O perpendicular to the vertical plane through the centroid, V the volume of water displaced, and z the distance between the centroid of solid and portion immersed, this is of course equivalent to $Ak^2 > Vz$.

The distances, SG_1, CR are investigated quite independently of one another, and therefore, *contrary to what at first sight might appear from inspection of the figure*, it is not *assumed* that SG_1 is either greater or less than CR.

7208. (By A. McMURCHY, B.A.)—If n be the length of a normal chord to a surface $ax^2+by^2+cz^2 = 1$ at a point for which r is the distance from the centre, and p the perpendicular on the tangent plane, prove that

$$\frac{2}{np^2} - \frac{a+b+c}{p^2} + bc + ca + ab - abcr^2 = 0.$$

Solution by the Rev. T. R. Terry, M.A.; Sarah Marks; *and others.*

With the given notation, we have

$$\frac{1}{p^2} = a^2x^2 + b^2y^2 + c^2z^2, \quad \gamma^2 = x^2 + y^2 + z^2, \quad 1 = ax^2 + by^2 + cz^2 \ldots (1, 2, 3).$$

To find n, we have

$$\frac{\xi - x}{ax} = \frac{\eta - y}{by} = \frac{\zeta - z}{cz} = np, \quad \text{therefore} \quad \xi = x(1 + anp),$$

therefore, substituting in the equation to the surface, we get

$$np = \frac{2(a^2x^2 + b^2y^2 + c^2z^2)}{a^2y^2 + b^2y^2 + c^2z^2}, \quad \text{or} \quad \frac{2}{np^3} = a^3x^2 + b^3y^2 + c^3z^2 \ldots\ldots\ldots (4).$$

Eliminating x^2, y^2, z^2 from (1, 2, 3), (4), we obviously have the required result.

6691. (By Professor Hudson, M.A.)—Prove (1) that, if the curve $r = a + b\cos\theta$ have a point of inflexion, a must lie between b and $2b$; and (2) trace the change in form of the curve, as a changes from 0 to $2b$.

Solution by Professor Scott, M.A.; J. O'Regan; *and others.*

The curve is the limaçon, the inverse of a conic with regard to the focus. If $a = 0$, the curve is a circle with diameter b; if $a = b$, it becomes the Cardioid; if $a < b$, it resembles the Cardioid with a loop inside the curve, which loop disappears when $a > b$. Forming the expression

$$0 = r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - \frac{r\,d^2r}{d\theta^2} = a^2 + 3ab\cos\theta + 2b^2,$$

we get $$-\cos\theta = \frac{a^2 + 2b^2}{3ab};$$

whence $a^2 + 2b^2 < 3ab$, or $a^2 + b^2 - 3ab \equiv (a - b)(a - 2b) < 0$,

whence $$a > b < 2b.$$

7197. (By the Editor)—With two given sides, construct a triangle such that the square of the sine of half the angle contained by the sides may be equal to the squares of the sines of the other two semi-angles.

I. *Solution by* W. H. Blythe, M.A.; Sarah Marks; *and others.*

Let a, b, be the given sides of the triangle, and let

$\sin^2\frac{1}{2}C = \sin^2\frac{1}{2}A + \sin^2\frac{1}{2}B$; then $1 + \cos C = \cos A + \cos B$,

$\therefore\ 2\cos^2\frac{1}{2}C = 2\cos\frac{1}{2}(A + B)\cos\frac{1}{2}(A - B)$, $\therefore\ \cos\frac{1}{2}(A - B) = \cot\frac{1}{2}C\cos\frac{1}{2}C$;

and, as $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{1}{2}C$, $\therefore \tan^2\frac{1}{2}C\sec^2\frac{1}{2}C = 1+\left(\frac{a-b}{a+b}\right)^2\cot^2\frac{1}{2}C$,

therefore $$\tan^6\tfrac{1}{2}C+\tan^4\tfrac{1}{2}C-\tan^2\tfrac{1}{2}C = \frac{(a-b)^2}{(a+b)^2}.$$

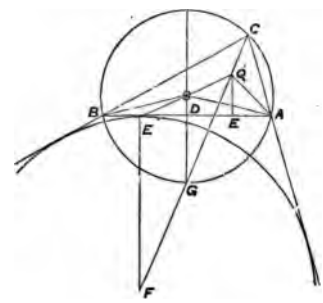
Hence, putting $3\tan^2\frac{1}{2}C = 4\cos\phi - 1$, we have

$$(a+b)^2(16\cos 3\phi + 11) = 27(a-b)^2;$$

whence we find 3ϕ, then ϕ, and from ϕ, $\frac{1}{2}C$; and the problem reduces to ordinary solution of triangles, a, b, and C given.

[See the solutions of the EDITOR'S Quest. 6287, on pp. 35—38 of Vol. 36 of the *Reprints*, where a construction is given by the aid of a parabola and a rectangular hyperbola.]

II. *Solution by* M. COLLINS, B.A.; J. O'REGAN; *and others.*



The triangle is of the form in part (3) of Quest. 6287. Thus, if c be the third or required side, Q the centre, r the radius of the inscribed circle, and O the centre of the circumscribed circle, we have $r(a+b+c) = ab\sin C$, which differentiated gives (since r is a maximum, and therefore $dr = 0$)

$$r\,dc = ab\cos C\,.\,dC;$$

also $$c^2 = a^2+b^2-2ab\cos C;$$

whence, differentiating,

$$c\,.\,d = ab\sin C\,.\,dC;$$

and from these equations we obtain

$$r = c\cot C = 2BD\,.\cot BOD = 2OD;$$

hence $DE = DE'$, therefore $E'F = 2OG$, that is, $r' =$ radius of circle escribed to side $c = 2R =$ diameter of circumscribed circle.

Now $R = \frac{abc}{4\Delta}$, and $r'(s-c) = \Delta$, therefore $2(s-c) = \frac{4\Delta^2}{b}$,

therefore $$abc(s-c) = 2\Delta^2 = 2s(s-a)(s-b)(s-c),$$

and as $c = 2s-a-b$, we get $2s^3-2(a+b)s^2+ab(a+b) = 0$, an equation which determines s, and therefore the construction of the triangle. It is easy to see that the line DQ, which joins the middle D of the base to the centre Q of the inscribed circle, would pass through the point of intersection of the three altitudes of the required $\triangle ABC$ when its inscribed circle (or its radius QE) is a maximum.

7224. (By Professor COCHEZ.)—Démontrer que la fonction

$$ke^{2x}\left\{[(x+y)^2+c]^{\frac{1}{2}}-x-y\right\} = [(x+y)^2+c]^{\frac{1}{2}}+x+y,$$

est telle que $x+y = y'$.

Solution by the Rev. J. L. Kitchin, M.A.; D. Edwardes; *and others.*

We have $$\frac{[(x+y)^2+c]^{\frac{1}{2}}+x+y}{[(x+y)^2+c]^{\frac{1}{2}}-x-y} = ke^{2x},$$

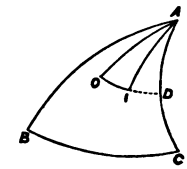
$$\therefore \quad \frac{(x+y)^2+c}{(x+y)^2} = \frac{(ke^{2x}+1)^2}{(ke^{2x}-1)^2}, \quad \therefore \quad x+y = \left(\frac{c}{4k}\right)^{\frac{1}{2}}(ke^x-e^{-x}),$$

therefore $$y'' = x+y.$$

6864. (By Professor Hudson, M.A.)—If I, O be the centres of the inscribed and circumscribed circles of a spherical triangle, r, R their radii, and $2s = a+b+c$, prove that $2 \sin s \cos IO = (\sin a + \sin b + \sin c) \cos r \cos R$.

Solution by G. M. Reeves, M.A.; Professor Matz, M.A.; *and others.*

Let O be the centre of the circumscribed, and I the centre of the inscribed circles of a spherical triangle ABC; then



$$\cos OI = \cos OA \cos IA + \sin OA \sin IA \cos OAI.$$

Now $OAC = S-B$ and $IAC = \frac{1}{2}A$,

therefore $$OAI = \tfrac{1}{2}(C-B).$$

Also $OA = R$, and ID perpendicular to AC is equal to r, and $AD = s-a$; therefore, from ΔaID,

$$\cos IA = \cos(s-a)\cos r \quad \text{and} \quad \sin r = \sin IA \sin \tfrac{1}{2}A,$$

therefore $$\cos OI = \cos R \cos r \cos(s-a) + \sin R \frac{\sin r}{\sin \frac{1}{2}A} \cos \tfrac{1}{2}(B-C)$$

$$= \left\{ \cos(s-a) + \frac{\cos \frac{1}{2}(B-C)}{\sin \frac{1}{2}A} \tan R \tan r \right\} \cos R \cos r$$

$$= \left\{ \cos(s-a) + \frac{\sin \frac{1}{2}(b+c)}{\sin \frac{1}{2}a} \cdot \frac{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}{n} \cdot \frac{n}{\sin s} \right\}$$

$$\times \cos R \cos r,$$

$$\therefore 2 \sin s \cos OI = [2 \sin s \cos(s-a) + 4 \sin \tfrac{1}{2}(b+c) \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c] \cos R \cos r$$

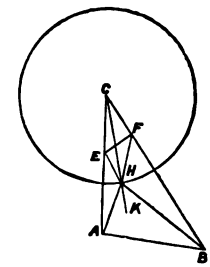
$$= [\sin(b+c) + \sin a + 2 \sin \tfrac{1}{2}(b+c) \{\cos \tfrac{1}{2}(b-c) - \cos \tfrac{1}{2}(b+c)\}] \cos R \cos r$$

$$= [\sin(b+c) + \sin a + \sin b + \sin c - \sin(b+c)] \cos R \cos r$$

$$= (\sin a + \sin b + \sin c) \cos R \cos r.$$

6878. (By T. C. Simmons, M.A.)—If A, B are two points in the plane of a given circle; give a geometrical construction for finding the position of a point P on the circumference, such that the lines AP, BP make equal angles with the tangent at P.

Solution by Dr. CURTIS; E. RUTTER; *and others.*



This, which is known as ALHAZEN'S problem, admits of the following geometrical solution:— If C be the centre of the circle and a its radius, take E and F so that $AC \cdot CE = a^2 = BC \cdot CF$; draw the hyperbola-locus of vertex of all triangles whose common base is EF, and the difference of whose base angles = $\angle CEF \sim CFE$; let H be any one of the four points in which this hyperbola cuts the circle; then, as

$$\angle EFH - FEH = CEF - CFE,$$

we have $\angle CEH = CFH$, and in the triangles ACH, HCE, we have $AC : CH = CH : CE$, therefore $\angle AHC = CEH$; similarly $\angle BHC = CFH$, and therefore $\angle AHC = BHC$, and $\angle AHK = BHK$, therefore H is one of the required points.

7215. (By A. H. CURTIS, LL.D., D.Sc.)—Prove that

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \&c. \ldots \textit{ad inf.} = \frac{\pi^4}{90},$$

$$\frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4} - \&c. \ldots \textit{ad inf.} = \frac{(2^3-1)\,\pi^4}{2^3 \times 90} = \frac{7\pi^4}{720},$$

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \&c. \ldots \textit{ad inf.} = \frac{2^4-1}{2^4} \times \frac{\pi^4}{90} = \frac{\pi^4}{96};$$

$$\frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \&c. \ldots \textit{ad inf.} = \frac{\pi^8}{3 \cdot 5 \cdot 7 \cdot 90} = \frac{\pi^8}{9450},$$

$$\frac{1}{1^8} - \frac{1}{2^8} + \frac{1}{3^8} - \&c. \ldots \textit{ad inf.} = \frac{2^7-1}{2^7} \times \frac{\pi^8}{9450} = \frac{127}{128} \times \frac{\pi^8}{9450},$$

$$\frac{1}{1^8} + \frac{1}{3^8} + \frac{1}{5^8} + \&c. \ldots \textit{ad inf.} = \frac{2^8-1}{2^8} \times \frac{\pi^8}{9450} = \frac{255}{256} \times \frac{\pi^8}{9450} = \frac{17\pi^8}{256 \times 630}.$$

Solution by the Rev. T. R. TERRY, M.A.; Professor MUKHOPÂDHYÂY, M.A.; *and others.*

Since $\dfrac{\sin\theta}{\theta} = \left(1 - \dfrac{\theta^2}{\pi^2}\right)\left(1 - \dfrac{\theta^2}{2^2\pi^2}\right)\ldots$, we have, by taking logs,

$$\log\left(1 - \frac{\theta^2}{3!} + \ldots\right) = -\left(\frac{\theta^2}{\pi^2} + \frac{1}{2}\,\frac{\theta^4}{\pi^4} + \ldots\right) - \left(\frac{\theta^2}{2^2\pi^2} + \ldots\right) - \ldots$$

Equating coefficients of θ^4 and θ^8, and denoting the series in the question by $S_1, S_2 \ldots$,

$$-\frac{1}{2\pi^4} S_1 = \frac{1}{5!} - \frac{1}{2}\,\frac{1}{3!\,3!}, \quad \text{therefore } S_1 = \frac{\pi^4}{90},$$

$$-\frac{1}{4\pi^8} S_4 = \frac{1}{9!} - \frac{1}{2}\left(\frac{1}{5!\,5!} + \frac{2}{3!\,7!}\right) + \frac{1}{3}\,\frac{3}{3!\,3!\,5!} - \frac{1}{4}\,\frac{1}{(3!)^4} = -\frac{1}{4}\,\frac{1}{9450},$$

therefore $$S_4 = \frac{\pi^3}{9450}.$$

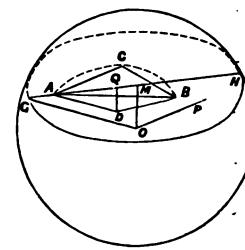
Also $$S_2 = S_1\left(1-2\frac{1}{2^4}\right), \quad S_3 = S_1\left(1-\frac{1}{2^4}\right),$$

$$S_5 = S_4\left(1-2\frac{1}{2^8}\right), \quad S_6 = S_4\left(1-\frac{1}{2^8}\right),$$

which give the required results.

6924. (By Professor Seitz, M.A.)—A triangle is formed by joining three random points within a sphere; prove that the average area (1) of the triangle is $\frac{8}{7}\pi r^2$, and (2) of its inscribed circle is $(\frac{2}{3}\frac{5}{5}\frac{5}{0}\frac{1}{0} - \frac{4}{3}\frac{8}{5}\log 2)\,\pi r^2$.

Solution by the Proposer.



Let A, B, C be three random points within the sphere whose centre is O, M the centre of the section of the sphere made by the plane through A, B, C, GH the diameter of this section through A, Q the point in GH equally distant from A and B, D the centre of the circle through A, B, C, and OP a radius such that AB is parallel to the plane MOP.

Let B be confined to the circle whose centre is M and whose circumference passes through A, and C to the circle whose centre is D and whose circumference passes through A and B; and put $OG = r$, $AM = x$, $AB = 2y$, $\angle BAM = \theta$, $\angle ADQ = \phi$, $\angle CAB = \psi$, $\angle GOM = \mu$, $\angle MOP = \nu$, the angle the plane MOP makes with a fixed plane through $OP = \omega$, and z = the radius of the inscribed circle of ABC, and u = area ABC. Then we have

$$AC = 2y \sin(\phi-\psi)\operatorname{cosec}\phi, \quad z = 2y \operatorname{cosec}\tfrac{1}{2}\phi \sin\tfrac{1}{2}\psi \sin\tfrac{1}{2}(\phi-\psi),$$

and $$u = 2y^2 \sin\psi \sin(\phi-\psi)\operatorname{cosec}\phi;$$

also an element of the sphere at A is $r\sin\mu\, d\mu \,.\, 2\pi x\, dx$, at B it is $8y^2\, dy\, d\theta\, d\nu$, and at C it is

$$d\,.\,(AC)\sin(\phi-\psi)\,.\,AC\sin\psi\sin\nu\, d\omega\,.\,2y\operatorname{cosec}\phi\, d\psi$$

$$= 8y^3 \operatorname{cosec}^4\phi \sin^2\psi \sin^2(\phi-\psi)\, d\phi\, d\psi \sin\nu\, d\omega.$$

The limits of θ are $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; of ϕ, 0 and $\frac{1}{2}\pi+\theta$, and doubled, since C may lie on either side of AB; of ψ, 0 and ϕ; of μ, 0 and $\frac{1}{2}\pi$; of x, 0 and $r\sin\mu = x'$, and the result multiplied by 6 to allow for the interchange of the points; of y, 0 and $x\cos\theta = y'$; of ν, 0 and π; and of ω, 0 and 2π.

The whole number of ways the three points can be taken is $(\frac{4}{3}\pi r^3)^3$.

1. The average area of the triangle is

$$\frac{12.3^3}{4^3\pi^3\gamma^6}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi+\theta}\int_0^{\phi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi'}\int_0^{y'}\int_0^{\pi}\int_0^{2\pi} u\,d\theta\,\operatorname{cosec}^4\phi\,d\phi\,\sin^2\psi\,\sin^2(\phi-\psi)\,d\psi\,r \times\sin\mu\,d\mu\,.\,2\pi x\,dx\,.\,64y^5\,dy\sin v\,dv\,d\omega$$

$$=\frac{5184}{\pi\gamma^6}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi+\theta}\int_0^{\phi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi'}\int_0^{y'} d\theta\,\operatorname{cosec}^5\phi\,d\phi\,\sin^3\psi\,\sin^3(\phi-\psi)\,d\psi \times\sin\mu\,d\mu\,x\,dx\,y^7\,dy$$

$$=\frac{324r^2}{5\pi}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi+\theta}\int_0^{\phi}\int_0^{\frac{1}{2}\pi}\cos^8\theta\,d\theta\,\operatorname{cosec}^5\phi\,d\phi\,\sin^3\psi\,\sin^3(\phi-\psi)\,d\psi\,\sin^{11}\mu\,d\mu$$

$$=\frac{9216r^2}{385\pi}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi+\theta}\int_0^{\phi}\cos^8\theta\,d\theta\,\operatorname{cosec}^5\phi\,d\phi\,\sin^3\psi\,\sin^3(\phi-\psi)\,d\psi$$

$$=\frac{192r^2}{385\pi}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi+\theta}(6\phi\cos\phi\,\operatorname{cosec}^3\phi-15\phi\cos\phi\,\operatorname{cosec}^5\phi+15\operatorname{cosec}^4\phi -11\operatorname{cosec}^2\phi)\cos^8\theta\,d\theta\,d\phi$$

$$=\frac{48r^2}{385\pi}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\left\{(\tfrac{1}{2}\pi+\theta)(15-12\cos^2\theta)+15\sin\theta\cos\theta-2\sin\theta\cos^3\theta\right\} \times\cos^4\theta\,d\theta=\tfrac{2}{7}\pi r^2.$$

2. The average area of the inscribed circle of the triangle is

$$\frac{12.3^3}{4^3\pi^3\gamma^6}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi+\theta}\int_0^{\phi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi'}\int_0^{y'}\int_0^{\pi}\int_0^{2\pi}\pi x^2 d\theta\,\operatorname{cosec}^4\phi\,d\phi\,\sin^2\psi\,\sin^2(\phi-\psi)\,d\psi \times r\sin\mu\,d\mu\,.\,2\pi x\,dx\,.\,64y^5\,dy\sin v\,dv\,d\omega$$

$$=\frac{10368}{\gamma^6}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi+\theta}\int_0^{\phi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi'}\int_0^{y'} d\theta\,\operatorname{cosec}^4\phi\,\operatorname{cosec}^2\tfrac{1}{2}\phi\,d\phi\,\sin^2\psi\,\sin^2\tfrac{1}{2}\psi \times\sin^2(\phi-\psi)\,\sin^2\tfrac{1}{2}(\phi-\psi)\,d\psi\,\sin\mu\,d\mu\,x\,dx\,y^7\,dy$$

$$=\frac{648r^2}{5}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi+\theta}\int_0^{\phi}\int_0^{\frac{1}{2}\pi}\cos^8\theta\,d\theta\,\operatorname{cosec}^4\phi\,\operatorname{cosec}^2\tfrac{1}{2}\phi\,d\phi\,\sin^2\psi\,\sin^2\tfrac{1}{2}\psi \times\sin^2(\phi-\psi)\,\sin^2\tfrac{1}{2}(\phi-\psi)\,d\psi\,\sin^{11}\mu\,d\mu$$

$$=\frac{18432r^2}{385}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi+\theta}\int_0^{\phi}\cos^8\theta\,d\theta\,\operatorname{cosec}^4\phi\,\operatorname{cosec}^2\tfrac{1}{2}\phi\,d\phi\,\sin^2\psi\,\sin^2\tfrac{1}{2}\psi \times\sin^2(\phi-\psi)\,\sin^2\tfrac{1}{2}(\phi-\psi)\,d\psi$$

$$=\frac{96r^2}{1925}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi+\theta}(60\phi+30\phi\cos 2\phi+15\phi\cos\phi\cos 2\phi-143\sin\phi +38\sin\phi\cos\phi+99\sin^3\phi)\cos^8\theta\,d\theta\,\operatorname{cosec}^4\phi\,\operatorname{cosec}^2\tfrac{1}{2}\phi\,d\phi$$

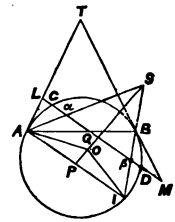
$$=\frac{96r^2}{1925}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\left\{(\tfrac{1}{2}\pi+\theta)(42\sin\theta-14\sin\theta\cos^2\theta+32\sin\theta\cos^4\theta-42+60\cos^2\theta) +42\cos\theta-92\cos^3\theta-42\sin\theta\cos\theta-32\sin\theta\cos^3\theta +45\tfrac{7}{15}\cos^5\theta+32\cos^5\theta\log[\tfrac{1}{4}(1-\sin\theta)]\right\}\cos^3\theta\,d\theta$$

$$=\pi r^2\left(\tfrac{2}{1}\tfrac{4}{9}\tfrac{4}{2}\tfrac{1}{0}-\tfrac{4}{3}\tfrac{4}{5}\log 2\right).$$

6917. (By J. L. McKenzie, B.A.)—A straight line XY cuts the tangents at two given points A and B on a given circle in L and M respec-

tively, and the circle itself in C and D; then (1) if the circles ABL, ABM cut XY in α and β respectively, prove $C\alpha = D\beta$; and (2) if the circles passing through L and M, and touching the given circle in B and A respectively, cut XY in α' and β', prove $C\alpha' = D\beta'$.

Solution by Professor HADAMARD; BELLE EASTON; *and others.*



1. Prolongeons $B\beta$ jusqu'à sa rencontre en I avec la circonférence et joignons AI, l'angle AIB est égal à l'angle LAB et par conséquent (puisque le quadrilatère LAB est inscriptible) à l'angle $\alpha\beta$B, ce qui montre que les droites $\beta\alpha$, IA sont parallèles. Mais alors les arcs CA, DI, ainsi que les cordes qui les sous-tendent, sont égaux, ainsi que les angles ACD, CDI. De plus l'angle $S\alpha\beta$, supplément de $A\alpha\beta$, est égal à ABI, supplément de ABM, puisque dans le quadrilatère inscrit $A\alpha$BM les angles $A\alpha$M, ABM, inscrits dans le même segment de cercle, sont égaux. Il est donc égal à $S\beta\alpha$, puisque $S\beta\alpha$ = TAB = TBA. Les triangles $CA\alpha$, $DI\beta$ sont donc égaux comme ayant un côté égal (CA = DI) et deux angles égaux (ACD = IDC, $\alpha = \beta$). Donc $C\alpha = D\beta$.

2. Le point L [the modified figure can be easily imagined] appartenant à la tangente commune aux cercles ABC, $M\alpha'\beta$ a même puissance par rapport à tous deux. On en tire

$$L\alpha' = \frac{LC.LD}{LM} \quad \text{et} \quad C\alpha' = \frac{LC(LM-LD)}{LM} = \frac{LC.MD}{LM},$$

expression dont la symétrie montre que $C\alpha' = D\beta'$.

7263. (By J. J. WALKER, M.A.)—Show that the principal axes through any point (O) of a homogeneous solid, with respect to which the bounding surface (S) is convex, are also principal axes for the shell (S) supposed to have at each point a surface density proportional to the perpendicular from (O) on the tangent plane at that point.

Solution by the Rev. T. R. TERRY, M.A.; Professor MATZ, M.A.; *and others.*

Draw radii vectores from O to each point P of the shell S, and in OP take a point Q so that $\frac{OQ}{OP} = \lambda$, and over the surface S′ formed by Q distribute uniformly a mass equal to that distributed uniformly over S. Then it is obvious that the products and moments of inertia round any axes through O for the shells S′ and S are in ratio $\lambda^2 : 1$.

Now split up the solid into an infinite number of shells like S′, and it follows at once that the products and moments of inertia round any axes through O for the solid are respectively one-third the corresponding products and moments for the shell S whose mass is the same as that of the solid and whose surface density at any point is proportional to the perpendicular from O on the tangent plane at that point.

7200 & 7233. (By G. S. Carr, B.A.)—(7200) If two non-intersecting circles, having radii a, b, centres A, B, and distance between the centres c, be inverted into concentric circles having their common centre at a distance d from A in the direction AB; show that α, β, the radii of the images, must have the values

$$\alpha = \frac{(2cd - a^2 + b^2 - c^2 + S)\,a}{a^2 - b^2 + c^2 - S}; \quad \beta = \pm \frac{(2cd - a^2 + b^2 - c^2 + S)\,b}{a^2 - b^2 - c^2 - S},$$

where $$S = \sqrt{(a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2)},$$

and the + or − sign of β is to be read according as the smallest circle, radius b, lies within or without the other.

(7233.) If two non-intersecting circles, having radii a, b, centres A, B, and distance between the centres c, be inverted into concentric circles with radii α, β, and centre O; show that, if we denote

$$(a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2)^{\frac{1}{2}}$$

by S, we have

$$AO = \frac{a^2 - b^2 - c^2 - S}{2c} + \frac{k^2 c}{S}, \quad \alpha = \frac{2k^2c^2a}{S(a^2 - b^2 + c^2 - S)}, \quad \beta = \frac{\pm 2k^2c^2b}{S(a^2 - b^2 - c^2 - S)},$$

a being $> b$ and the + or − sign being read according as the smallest circle is within or without the largest.

Solution by the Rev. T. R. Terry, M.A.; Prof. Nash, M.A; *and others.*

If C be the centre of inversion and $CA = p$, $CB = q$, we have

$$\frac{pk^2}{p^2 - a^2} = \frac{qk^2}{q^2 - b^2} = p - d, \quad \alpha = \frac{ak^2}{p^2 - a^2}, \quad \beta = \frac{bk^2}{q^2 - b^2}, \quad p - q = c \ldots (1, 2, 3, 4).$$

Substituting for (4) in (1), we have

$$cp^2 - p(a^2 - b^2 + c^2) + a^2c = 0, \quad \therefore\ p = \frac{a^2 - b^2 + c^2 - S}{2c}, \quad \therefore\ q = \frac{a^2 - b^2 - c^2 - S}{2c},$$

which gives the values α and β in Quest. 7233. Also

$$AO = p - \frac{pk^2}{p^2 - a^2} = \frac{a^2 - b^2 + c^2 - S}{2c} + \frac{k^2 c}{S}.$$

Again, from (1), substituting values of p and $p^2 - a^2$, we get

$$k^2 = \frac{(2cd - a^2 + b^2 - c^2 + S)\,S}{2c^2},$$

which, by substitution, gives the values of α and β Quest. 7200.

6879. (By J. O'Regan.)—The horizontal base AB of a triangle being given in magnitude and position, and the side AC in magnitude only, construct the triangle so that a body may fall along CB from C to B in the least time possible.

Solution by Dr. CURTIS; G. M. REEVES, M.A.; *and others.*

This is a particular case of the following well-known problem—To find the line of quickest, or slowest, descent from a circle to a point, or *vice versâ*, both being situated in the same vertical plane. Let B (Fig. 1) be the given point, and A and a the centre and radius of the given circle. Consider the time, t, of falling down any distance BC; produce BC to cut the circle in D; draw DE vertically and BE horizontally; denote BC, BD, DE by s, l, h, respectively; then $t^2 = \frac{2ls}{gh}$, but $ls\,(= AB^2 - a^2)$ is known; and therefore, when t is minimum or maximum, h is maximum or minimum, therefore D is one of the extremities of the vertical diameter. If the point B lie between the two horizontal tangents, there are two solutions for t minimum, the horizontal line through B corresponding to maximum time, ∞; when the point B lies on the same side of both horizontal tangents, one solution gives t maximum, the other minimum.

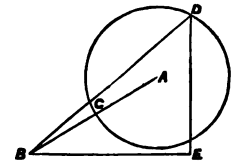


Fig 1.

In the case proposed, denoting by α and β the angles BAC (Fig. 2) and ACD or ADC, we have

$$\frac{\pi}{2} + \alpha = 2\beta, \text{ and } \therefore\ \alpha = 2\tan^{-1}\left(\frac{BA}{AD}\right) - \frac{\pi}{2}.$$

It can similarly be proved that, two circles being situated in the same vertical plane, the lines of quickest, or slowest, descent from one to the other will be the intercepts on the lines joining the extremities of the two vertical diameters of the circles.

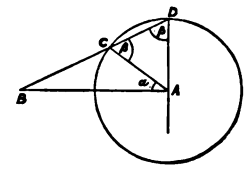


Fig. 2.

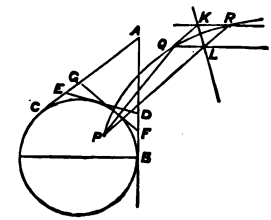
6817. (By T. C. SIMMONS, M.A.)—From Question 6720 deduce, by reciprocation, a property of the parabola, and give an independent proof of the theorem thus obtained.

Solution by the PROPOSER; E. RUTTER; *and others.*

Reciprocate the circle into a parabola whose focus is B (of course B being the centre of reciprocation).

Let P, Q, R correspond with lines AC, DE, FG; then RK, parallel to axis of parabola, will be the polar of F, and QL the polar of D. Also the chord PQ will be polar of E, and PR of G.

Hence K, the point of intersection of PQ and RK, will be the pole of EF, and similarly L the pole of DG; therefore KL is the polar of the intersection of DG and EF. But this point of intersection



lies on BC, therefore KL is perpendicular to BC. But the tangent at P is perpendicular to BC, therefore KL is parallel to the tangent at P.

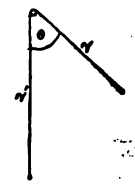
[Hence we obtain the following theorem :—PQ, PR are any two chords of a parabola. If PQ meet the diameter through R in K, and PR meet the diameter through Q in L, then KL is parallel to the tangent at P].

6865. (By G. F. Walker, M.A.)—Two equal particles are connected by a string which passes over a smooth peg; prove that, if one move vertically, and θ be measured from the vertical downwards, and u be the reciprocal of the radius vector, the equation of motion of the other is

$$\frac{\sin\theta}{u^3}+\frac{d}{d\theta}\left\{\frac{\sin^2\tfrac{1}{2}\theta+\frac{\sin\theta}{u}\frac{du}{d\theta}}{u^2\left(2\frac{d^2u}{d\theta^2}+u\right)}\right\}=0.$$

Solution by G. M. Reeves, M.A.; G. Heppel, M.A.; *and others.*

Measuring θ upwards, instead of downwards as in question, and taking mass of each particle to be unity, the equations of motion are



$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=g\cos\theta-\mathrm{T},\quad \frac{1}{r}\cdot\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=-g\sin\theta,$$

and

$$\frac{d^2r'}{dt^2}=g-\mathrm{T},$$

where T is tension of the string. Eliminating T, and observing that $r+r'$ is constant, we get

$$2\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=-g\,(1-\cos\theta)=-2g\sin^2\tfrac{1}{2}\theta,$$

and

$$\frac{1}{r}\cdot\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=-g\sin\theta.$$

Let $r=\frac{1}{u}$, and $h=r^2\frac{d\theta}{dt}$, so that $\frac{d\theta}{dt}=hu^2$. These equations become

$$2\frac{dh}{dt}\cdot\frac{du}{d\theta}+2h^2u^2\frac{d^2u}{d\theta^2}+h^2u^2=2g\sin^2\tfrac{1}{2}\theta,\text{ and }u\frac{dh}{dt}=-g\sin\theta,$$

whence

$$h^2=2g\frac{\sin^2\tfrac{1}{2}\theta+\frac{\sin\theta}{u}\frac{du}{d\theta}}{u^2\left(2\frac{d^2u}{d\theta^2}+u\right)},$$

and

$$\frac{\sin\theta}{u^3}+\frac{d}{d\theta}\left\{\frac{\sin^2\tfrac{1}{2}\theta+\frac{\sin\theta}{u}\cdot\frac{du}{d\theta}}{u^2\left(2\frac{d^2u}{d\theta^2}+u\right)}\right\}=0.$$

The limits of θ are $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; of ϕ, 0 and $\frac{1}{2}\pi+\theta$, and doubled, since P may lie on either side of MN; of ψ, 0 and ϕ; of x, 0 and r; and of y, 0 and $x\cos\theta = y'$. Hence, since the whole number of ways the three points are taken is $\frac{1}{6}\pi^3 r^6$, the mean value of z is

$$\frac{12}{\pi^3 r^6}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi+\theta}\int_0^{\phi}\int_0^{r}\int_0^{y'} z\operatorname{cosec}^3\phi\, d\theta\, d\phi\,.\,4\sin\psi\sin(\phi-\psi)\,d\psi\,.\,2\pi x\,dx\,.\,4y^3 dy$$

$$= \frac{768r}{35\pi^2}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi+\theta}\int_0^{\phi} \cos^5\theta\, d\theta \operatorname{cosec}^3\phi \operatorname{cosec}\tfrac{1}{2}\phi\, d\phi \sin\tfrac{1}{2}\psi \sin\tfrac{1}{2}(\phi-\psi)\sin\psi \times \sin(\phi-\psi)\,d\psi$$

$$= \frac{64r}{35\pi^2}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi+\theta} (3\phi\cos\phi\cos\tfrac{1}{2}\phi - 6\sin\tfrac{1}{2}\phi + 14\sin^3\tfrac{1}{2}\phi)\cos^5\theta\, d\theta \times \operatorname{cosec}^3\phi \operatorname{cosec}\tfrac{1}{2}\phi\, d\phi$$

$$= \frac{64r}{105\pi^2}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\Big\{3(\tfrac{1}{2}\pi+\theta)(\sin^3\theta-1) + 9\cos\theta + 3\sin\theta\cos\theta - 5\cos^3\theta - 3\cos^3\theta\log\Big(\frac{1-\sin\theta}{2}\Big)\Big\}\cos^2\theta\, d\theta$$

$$= \frac{192r}{35\pi^2}\Big(\frac{128}{135} - \frac{\pi^2}{12}\Big).$$

7071. (By T. C. Simmons, M.A.)—Prove the two following constructions for finding the vertical chord of curvature at any point P of a parabola; (1) on PA produced take AR equal to three times the projection of the abscissa of P on AP; (2) from the focal distance SP cut off SV = AS, and on PS produced take SW = 3PS: draw a circle through the points A, V, W, meeting PA produced in R. In each of these cases, PR will be the required chord of curvature.

Solution by the Proposer; D. Eastwood, M.A.; *and others.*

1. Draw NL perpendicular to AP; and tangent PYT. Let PR be vertical chord of curvature. Then (as in the published Solutions to Besant's *Geometrical Conics*) we can show that

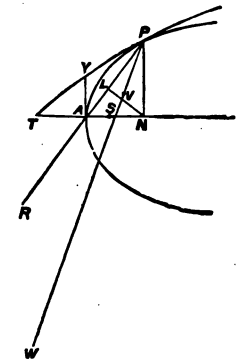
$$PA\,.\,PR = 4PY^2 = PT^2,$$

therefore $PA\,.\,AR + PA^2 = PN^2 + NT^2$,

$$PA\,.\,AR + PN^2 + AN^2 = PN^2 + 4AN^2,$$

$$PA\,.\,AR = 3AN^2 = 3PA\,.\,AL,$$

therefore $AR = 3AL$.



2. $$PA\,.\,PR = 4TY^2 = 4ST\,.\,TA = 4SP\,.\,PV$$

(since, if SV = AS, PV = AT)

$$= PW\,.\,PV;$$

therefore a circle passes through the points V, W, A, and R.

6883. (By T. Morley, L.C.P.)—If the water-section of a ship be taken for the base of a right cylinder equal to the volume of the ship, and H be the height of the cylinder; prove that the vertical oscillations of the vessel are the same as the vibrations of a simple pendulum whose length is SH; S being the specific gravity of ship and cargo.

Solution by Dr. Curtis; G. M. Reeves, M.A.; *and others.*

M being the mass of the cylinder, A its principal section, and ξ its depression, we have

$$M\frac{d^2\xi}{dt^2} = -Ag\xi, \quad \text{or} \quad SHA\frac{d^2\xi}{dt^2} + Ag\xi = 0, \quad \text{or} \quad \frac{d^2\xi}{dt^2} + \frac{g}{SH}\xi = 0;$$

comparing this with the equation of motion of a simple pendulum

$$\frac{d^2\xi}{dt^2} + \frac{g}{l}\xi = 0,$$

we get $l = SH$. It is not difficult to extend this problem to the case where the reservoir in which the ship is immersed is, or may be considered, a cylinder whose section, A + B, is so small that a depression of the ship causes an elevation of the water in the reservoir. The solution then is $l = \left(\frac{B}{A+B}\right) SH$.

6731. (By W. J. C. Sharp, M.A.)—If any number of particles attract each other with a force varying directly as their masses, and inversely as the cube of the distances between them; prove that

$$\Sigma m(x^2 + y^2 + z^2) = at^2 + bt + c,$$

whatever the origin and axes may be.

Solution by D. Edwardes; Professor Seitz, M.A.; *and others.*

Denoting by Q half the sum of the products, two and two, of the masses divided by the squares of their respective distances, the equations of motion for a particle (m) at x, y, z are

$$m\frac{d^2x}{dt^2} = \frac{dQ}{dx}, \quad m\frac{d^2y}{dt^2} = \frac{dQ}{dy}, \quad m\frac{d^2z}{dt^2} = \frac{dQ}{dz},$$

then

$$\tfrac{1}{2}\Sigma m\frac{d^2r^2}{dt^2} = \Sigma m\left(x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + z\frac{d^2z}{dt^2}\right) + \Sigma m\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}$$

$$= \Sigma\left(x\frac{dQ}{dx} + y\frac{dQ}{dy} + z\frac{dQ}{dz}\right) - 2\Sigma\int\frac{mm'}{f^3}\,df,$$

if f be the distance between two particles m, m'. But, by the nature of the function Q, $\quad \Sigma\left(x\frac{dQ}{dx} + y\frac{dQ}{dy} + z\frac{dQ}{dz}\right) = -\Sigma\frac{mm'}{f^2}$.

Hence $\Sigma m\frac{d^2r^2}{dt^2} = 2a$, and therefore, &c.

7037. (By the EDITOR.)—A particle moves on the surface of an ellipsoid under the action of an attractive force varying inversely as the cube of the distance from the plane containing the greatest and least axis; show that, by a proper arrangement of the initial circumstances, the path will be one of the circular sections of ellipsoid passing through the centre.

Solution by W. H. BLYTHE, M.A.; E. RUTTER; *and others.*

Take the equation to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, the attraction on the particle μy^{-3} parallel to axis y, taking the usual notation, the equations of motion are

$$\frac{a^2}{x}\cdot\frac{d^2x}{dt^2}=\frac{c^2}{z}\cdot\frac{d^2z}{dt^2}=\frac{b^2}{y}\left(\frac{d^2y}{dt^2}+\mu y^{-3}\right)=\frac{R}{\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{1}{2}}},$$

the mass of the particle taken as unity.

If the particle move in a central circular section, we have

$$x\left(\frac{1}{b^2}-\frac{1}{a^2}\right)^{\frac{1}{2}}=\pm z\left(\frac{1}{c^2}-\frac{1}{b^2}\right)^{\frac{1}{2}};$$

it is evident that this equation cannot co-exist with the above unless $\frac{d^2x}{dt^2}=0=\frac{d^2y}{dt^2}$, and therefore $\frac{d^2y}{dt^2}+\mu y^{-3}=0$. Solving these equations,

$$x=na\left(\frac{b^2-c^2}{a^2-c^2}\right)^{\frac{1}{2}}[\cos\theta+a_1],\quad z=nc\left(\frac{a^2-b^2}{a^2-c^2}\right)^{\frac{1}{2}}[\cos\theta+a_1],\quad y=nb\sin\theta,$$

where $\cos\theta=mt+p$, and $\mu=m^2n^2b^4$. Substituting these values in the equation to the ellipsoid, noticing that they must be true for all values of θ, we obtain $a_1=0$ and $n=1$; and, taking

$$\left(\frac{ds}{dt}\right)^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2=b^2\left(\frac{d\theta}{dt}\right)^2$$

from the above equations, therefore $\frac{ds}{d\theta}=b$; integrating from $\theta=0$ to 2π, we obtain $s=2b\pi$.

4387. (By Professor CAYLEY, F.R.S.)—Using the term "Cassinian" to denote a bi-circular quartic having four foci in a right line; show that the equation of a Cassinian having for its four foci the points $x=a$, $x=b$, $x=c$, $x=d$ on the axis of x, may be written in the four equivalent forms

$$\begin{vmatrix} . & \tau(d-c), & \sigma(b-d), & \rho(c-b) \\ \tau(c-d), & . & \rho(d-a), & \sigma(a-c) \\ \sigma(d-b), & \rho(a-d), & . & \tau(b-a) \\ \rho(b-c), & \sigma(c-a), & \tau(a-b), & . \end{vmatrix} (A^{\frac{1}{2}}, B^{\frac{1}{2}}, C^{\frac{1}{2}}, D^{\frac{1}{2}})=0,$$

that is, $\tau(d-c)\,B^{\frac{1}{2}} + \sigma(b-d)\,C^{\frac{1}{2}} + \rho(c-b)\,D^{\frac{1}{2}} = 0,$

$\tau(c-d)\,A^{\frac{1}{2}} \quad . \quad + \rho(d-a)\,C^{\frac{1}{2}} + \sigma(a-c)\,D^{\frac{1}{2}} = 0,$

&c., &c.,

where $A^{\frac{1}{2}}$, $B^{\frac{1}{2}}$, $C^{\frac{1}{2}}$, $D^{\frac{1}{2}}$ are the distances from the four foci respectively, and the parameters ρ, σ, τ are connected by the equation

$$\rho^2(a-d)(b-c) + \sigma^2(b-d)(c-a) + \tau^2(c-d)(a-b) = 0.$$

Show also that the curve has, at right angles to the axis of x, two double tangents, the equation whereof is any one of the three equivalent forms

$$(a+d-2x)(b+c-2x) : (b+d-2x)(c+a-2x) : (c+d-2x)(a+b-2x) = \rho^2 : \sigma^2 : \tau^2.$$

Solution by W. J. C. Sharp, M.A.

If the Cassinian be inverted about a focus, say the one whose abscissa is a, the inverse is a Cartesian, whose axial foci are the points inverse to the other three foci. If the radius of inversion be unity, and r, r_1, r_2 the three focal distances of any point on the Cartesian; its foci are the points on the axis at distances $\frac{1}{b-a}$, $\frac{1}{c-a}$, $\frac{1}{d-a}$ from the centre of inversion. Then (see Quest. 4667)

$$r = \frac{B^{\frac{1}{2}}}{A^{\frac{1}{2}}} \cdot \frac{1}{b-a}, \quad r_1 = \frac{C^{\frac{1}{2}}}{A^{\frac{1}{2}}} \cdot \frac{1}{c-a}, \quad r_2 = \frac{D^{\frac{1}{2}}}{A^{\frac{1}{2}}} \cdot \frac{1}{d-a}.$$

But, by Quest. 6177 (*Reprint*, Vol. xxxiii., p. 85),

$$\tau\frac{c-d}{(c-a)(d-a)} + \rho r_1 - \sigma r_2 = 0, \quad \sigma\frac{d-b}{(b-a)(d-a)} - \rho r + \tau r_2 = 0,$$

$$\rho\frac{b-c}{(b-a)(c-a)} + \sigma r - \tau r_1 = 0,$$

which lead, by substitution for r, r_1, r_2, to the three last of the equations in the question.

From these

$$\tau\rho(a-d)(d-c)\,B^{\frac{1}{2}} + \sigma\rho(d-a)(d-b)\,C^{\frac{1}{2}} + [\sigma^2(a-c)(d-b) + \tau^2(b-a)(d-c)]\,D^{\frac{1}{2}} = 0,$$

and $$\tau\sigma(c-a)(d-c)\,B^{\frac{1}{2}} + [\rho^2(d-a)(b-c) + \tau^2(a-b)(d-c)]\,C^{\frac{1}{2}} + \rho\sigma(a-c)(b-c)\,D^{\frac{1}{2}} = 0,$$

which are equivalent to the first equation, and the equation of condition.

For a double tangent perpendicular to the axis of x, the values of x at both points of contact must be equal, and the equation to the curve must give equal values for y^2, and consequently for x^2+y^2.

Now, one form of the equation is

$$\rho(b-c)\,A^{\frac{1}{2}} + \sigma(c-a)\,B^{\frac{1}{2}} + \tau(a-b)\,C^{\frac{1}{2}} = 0,$$

or, if $x^2+y^2 = r$, and $A = r+\alpha$, $B = r+\beta$, $C = r+\gamma$,

$$\rho(b-c)(r+\alpha)^{\frac{1}{2}} + \sigma(c-a)(r+\beta)^{\frac{1}{2}} + \tau(a-b)(r+\gamma)^{\frac{1}{2}} = 0,$$

the discriminant of which will be the equation to the two bitangents. This will be the eliminant of

$$\frac{\rho(b-c)}{(r+\alpha)^{\frac{1}{2}}} + \frac{\sigma(c-a)}{(r+\beta)^{\frac{1}{2}}} + \frac{\tau(a-b)}{(r+\gamma)^{\frac{1}{2}}} = 0,$$

and $$\frac{\rho(b-c)\alpha}{(r+\alpha)^2}+\frac{\sigma(c-a)\beta}{(r+\beta)^2}+\frac{\tau(a-b)\gamma}{(r+\gamma)^2}=0,$$

or $$\frac{\rho^2(b-c)^2}{\beta-\gamma}+\frac{\sigma^2(c-a)^2}{\gamma-\alpha}+\frac{\tau^2(a-b)^2}{\alpha-\beta}=0,$$

or $$\frac{\rho^2(b-c)}{b+c-2x}+\frac{\sigma^2(c-a)}{c+a-2x}+\frac{\tau^2(a-b)}{a+b-2x}=0,$$

and, by the equation of condition,

$$\rho^2(b-c)\left\{\frac{c-d}{b+c-2x}-\frac{a-d}{a+b-2x}\right\}+\sigma^2(c-a)\left\{\frac{c-d}{c+a-2x}-\frac{b-d}{a+b-2x}\right\}=0,$$

or $$\frac{(b+d-2x)(c+a-2x)}{\sigma^2}=\frac{(a+d-2x)(b+c-2x)}{\rho^2},$$

which, by symmetry, proves the given form of the equation.

7269. (By Belle Easton.)—One of two bags contains 10 sovereigns, and the other 10 shillings. One coin is taken out of each and placed in the other. This is repeated ten times. Find now the expectation of each bag.

Solution by the Rev. T. R. Terry, M.A.; J. O'Regan; *and others.*

Let u_n be the value in shillings of the sovereign bag after n operations; then $u_{n+1}=\frac{9}{10}u_n+\frac{1}{10}(210-u_n)=\frac{4}{5}u_n+21$, therefore $u_n=A(\frac{4}{5})^n+105$. But, when $n=0$, $u_n=200$, therefore $A=95$; therefore the values of the bags after n operations are

$$105+95.(\tfrac{4}{5})^n, \text{ and } 105-95.(\tfrac{4}{5})^n \text{ respectively.}$$

4897. (By the late Professor Clifford, F.R.S.)—Let U, V=0 be the point-equations, and u, $v=0$ the line-equations of the same two conics. If a tangent to U and a tangent to V are conjugate in respect of $u\pm\lambda v=0$, they will intersect on $U-\lambda^2V=0$. This last conic passes through the points of contact of the conics $u\pm\lambda v=0$ with the common tangents of u and v.

Solution by W. J. C. Sharp, M.A.

Referring the conics to their common self-conjugate triangle, let

$$U\equiv ax^2+by^2+cz^2, \quad V=a'x^2+b'y^2+c'z^2,$$

then, if $A=bc$, &c., we have

$$u=A\alpha^2+B\beta^2+C\gamma^2, \text{ and } v=A'\alpha^2+B'\beta^2+C'\gamma^2;$$

also let $\alpha_1 x+\beta_1 y+\gamma_1 z=0$ and $\alpha_2 x+\beta_2 y+\gamma_2 z=0$,
be the two tangents, then,

$$A\alpha_1^2+B\beta_1^2+C\gamma_1^2=0,\quad A'\alpha_2^2+B'\beta_2^2+C'\gamma_2^2=0,$$

and $$(A\pm\lambda A')\alpha_1\alpha_2+(B\pm\lambda B')\beta_1\beta_2+(C\pm\lambda C')\gamma_1\gamma_2=0.$$

Now (SALMON'S *Conics*, 5th ed., p. 258)

$$(A\alpha_1^2+B\beta_1^2+C\gamma_1^2)(A\alpha_2^2+B\beta_2^2+C\gamma_2^2)-(A\alpha_1\alpha_2+B\beta_1\beta_2+C\gamma_1\gamma_2)^2$$
$$\equiv\Delta\,[a(\beta_1\gamma_2-\beta_2\gamma_1)^2+b(\gamma_1\alpha_2-\gamma_2\alpha_1)^2+c(\alpha_1\beta_2-\alpha_2\beta_1)^2]\ldots(1),$$

and $$(A'\alpha_1^2+B'\beta_1^2+C'\gamma_1^2)(A'\alpha_2^2+B'\beta_2^2+C'\gamma_2^2)-(A'\alpha_1\alpha_2+B'\beta_1\beta_2+C'\gamma_1\gamma_2)^2$$
$$\equiv\Delta'[a'(\beta_1\gamma_2-\beta_2\gamma_1)^2+b'(\gamma_1\alpha_2-\gamma_2\alpha_1)^2+c'(\alpha_1\beta_2-\alpha_2\beta_1)^2]\ldots\ldots(2).$$

Multiplying (2) by λ^2 and subtracting the result from (1), and remembering the equations satisfied by $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$,

$$0=(\Delta a-\lambda^2\Delta' a')(\beta_1\gamma_2-\beta_2\gamma_1)^2+(\Delta b-\lambda^2\Delta' b')(\gamma_1\alpha_2-\gamma_2\alpha_1)^2$$
$$+(\Delta c-\lambda^2\Delta' c')(\alpha_1\beta_2-\alpha_2\beta_1)^2,$$

or $$0=\Delta(ax^2+by^2+cz^2)-\lambda^2\Delta'(a'x^2+b'y^2+c'z^2),$$

if (x, y, z) be the intersection of the two tangents, which therefore lies on $\Delta U-\lambda^2\Delta' V=0$, which reduces to the given form if $a'b'c'$ be determined so that $\Delta=\Delta'$. If $\alpha x+\beta y+\gamma z=0$ be a common tangent to u and v, $A\alpha^2+B\beta^2+C\gamma^2=0$ and $A'\alpha^2+B'\beta^2+C'\gamma^2=0$, and the point of contact with $u\pm\lambda v=0$ is $(A\pm\lambda A')\alpha:(B\pm\lambda B')\beta:(C\pm\lambda C')\gamma$, which will be found to satisfy the equation $\Delta U-\lambda^2\Delta' V=0$, in virtue of the equations satisfied by $\alpha\beta\gamma$.

7285. (By the EDITOR.)—From every point in a given curve a straight line is drawn, making with the ordinate on one side the same angle that the tangent makes with the ordinate on the other side: find (1) the locus of the intersections of these lines; and (2) prove that, if the given curve is the common parabola, the locus will be a parabola whose latus-rectum is triple that of the first.

Solution by W. H. BLYTHE, M.A.; Dr. CURTIS; *and others.*

1. The angle which the line drawn as in the question makes with the axis of x is $\tan^{-1}\left(-\frac{dy}{dx}\right)$. If (ξ, η) be the current coordinates, the equation to the line is

$$y-\eta+\frac{dy}{dx}(x-\xi)=0,$$

where x, y, and $\frac{dy}{dx}$ are related to each other as in the equation to the curve. The envelope of this line is the locus (1).

2. If $y^2=4ax$, we have $y=2a^{\frac{1}{2}}x^{\frac{1}{2}}$, and $\frac{dy}{dx}=\frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$, and the equation to the line becomes $a^{\frac{1}{2}}x^{-\frac{1}{2}}\xi+\eta=3a^{\frac{1}{2}}x^{\frac{1}{2}}$, or, putting $a^{-\frac{1}{2}}x^{\frac{1}{2}}=m$, $\eta=m\xi-\frac{3a}{m}$, the envelope of which is $\eta^2=-12a\xi$, which proves (2).

7280. (By Professor MALET, F.R.S.)—If any three conjugate semi-diameters of the ellipsoids

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{k^2}, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{k'^2},$$

meet the three surfaces respectively in the points P, Q, R, prove that the envelope of the plane PQR is the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{1+k^2+k'^2}.$$

Solution by Rev. T. R. TERRY, M.A.; J. P. JOHNSTONE; *and others.*

Take the three conjugate diameters as axes. Let the intercepts on these axes made by the first ellipsoid be A, B, C, by the second $\frac{A}{k}, \frac{B}{k}, \frac{C}{k}$, and by the third $\frac{A}{k'}, \frac{B}{k'}, \frac{C}{k'}$. Then the equation to the plane PQR is

$$\frac{x}{A}+\frac{ky}{B}+\frac{k'z}{C}=1 \quad \text{.....(1),}$$

and the equation to a fourth ellipsoid similar to the other three, and having semi-axes $\frac{a}{\lambda}, \frac{b}{\lambda}, \frac{c}{\lambda}$, is

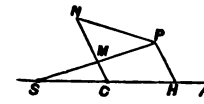
$$\frac{\lambda^2x^2}{A^2}+\frac{\lambda^2y^2}{B^2}+\frac{\lambda^2z^2}{C^2}=1 \quad \text{.....(2).}$$

But (1) will touch (2) if $\lambda^2=1+k^2+k'^2$, which proves the required result.

7299. (By R. KNOWLES.)—P is any point on an ellipse of which S and H are the foci; circles are drawn on SP and HP as diameters; show that (1) the loci of their centres are two equal ellipses intersecting on the minor axis of the given ellipse; (2) the circles touch the auxiliary circle; (3) the line joining the points of contact touches the ellipse at P.

Solution by ARTHUR COHEN, Q.C., M.P.; C. BICKERDIKE; *and others.*

1. Draw CMN parallel to HP, meeting SP in M, and the auxiliary circle in N, so that CN = CA the semi-major axis; then CM = ½HP, SM = ½SP, therefore



$$CM+SM=\tfrac{1}{2}(SP+HP)=CA=CN,$$

therefore NM = SM = MP, and the circle on SP as diameter touches the auxiliary circle at N.

2. Since SM + CM = CA, the locus of M is an ellipse with S and C as foci and major axis = CA. Similarly the locus of the middle point of HP is an ellipse with H and C as foci and major axis = CA.

3. Let O be the point where the two equal ellipses meet. Then we have

$$SO + CO = CA = HO + CO, \text{ therefore } SO = HO,$$

therefore O is a point on the perpendicular bisector of SH or on the minor axis of the ellipse. Again, since NM = MP, NP makes equal angles with NM and MP, or with HP and PS, and is therefore the tangent at P. Hence the line joining the point of contact with the auxiliary circle is the tangent at P.

7277. (By Professor SYLVESTER, F.R.S.)—If S_k denotes the sum of the k-ary products of 1, $\frac{1}{2}$, $\frac{1}{3}$, ... $\frac{1}{i-1}$, prove that the coefficient of any positive $(1+k)^{\text{th}}$ power of $\frac{1}{x}$, in the development of $\frac{1}{(1-e^{-x})^i}$ in a series of ascending powers of x, is $\Pi k \,.\, S_k$.

Solution by the PROPOSER.

Let Rx, $R_1 x$ in general denote certain series containing only positive powers of x, then

$$\frac{1}{(1-e^{-x})^i} = \left(1-\frac{d}{dx}\right)\left(1-\frac{1}{2}\frac{d}{dx}\right)\ldots\ldots\left(1-\frac{1}{i-1}\frac{d}{dx}\right)\left(\frac{1}{1-e^{-x}}\right)$$

$$= \left\{1 - S_1\frac{d}{dx} + S_2\left(\frac{d}{dx}\right)^2 \ldots\ldots\right\}\left\{\frac{1}{x} + Rx\right\}$$

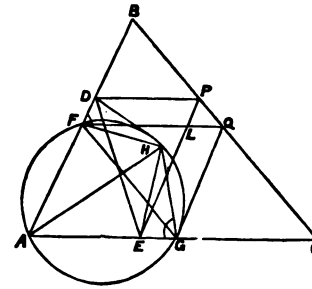
$$= \frac{1}{x} + \frac{S_1}{x^2} + \frac{1.2S_2}{x^3} + \frac{1.2.3S}{x^4} + \ldots + \frac{1.2\ldots(i-1)\,S_{i-1}}{x^i} + R_1 x.$$

Hence the theorem is proved.

It will be observed that the gist of the proof consists in separating the series for $\frac{1}{(1-e^{-x})^i}$ into i separate series, each containing a single negative power of x, the solitary and isolated sentinel as it were of an infinite host of terms.

7246. (By Capt. MACMAHON, R.A.)—From any point P in the side BC of the triangle ABC, lines PD, PE are drawn parallel to AC and AB, meeting those sides in D and E; prove that the envelope of DE is a parabola, whose focus is at a distance from A equal to $bc\,[2(b^2+c^2)-a^2]^{-\frac{1}{2}}$.

Solution by Dr. CURTIS; C. MORGAN, M.A.; *and others.*



From Q, the mid-point of BC, draw QF, QG parallel respectively to AC, AB; through F, G, A draw a circle, and find on it the point H, such that FH GH = FA : GA; then, drawing DH, EH, we have

$$DF : EG = PL : LQ = FA : GA = FH : GH;$$

therefore $DF : FH = EG : GH$,

and $\angle DFH = EGH$;

hence DFH, EGH are similar triangles and FDH = GEH; therefore the known point, H, is on the circle circumscribing the triangle ADE; if therefore from the point H perpendiculars be drawn on the fixed lines AB, AC, and the variable line DE, the feet of these three perpendiculars will lie on the fixed straight line joining the two former. The envelope required is therefore the envelope of the straight line which is perpendicular to any variable line passing through a given point at the point in which this line intersects a given fixed line, viz., a parabola whose focus is at H, and whose tangent at vertex is the line which joins the feet of the perpendiculars let fall on AB, AC, from the point H. It is not difficult to prove that

$$AH = bc\,[2\,(b^2+c^2)-a^2]^{-\frac{1}{2}}.$$

7232. (By R. F. SCOTT, M.A.)—If (a, b, c) stand for the straight line $ax+by+cz=0$, show that the six lines obtained by permutating a, b, c in (a, b, c) in all possible ways, all touch the conic

$$\frac{x^2+y^2+z^2}{a^2+b^2+c^2} - \frac{2\,(yz+zx+xy)}{a^2+b^2+c^2-2bc-2ca-2ab} = 0.$$

Solution by G. F. WALKER, M.A.; Professor MATZ, M.A.; *and others.*

The line $ax+by+cz=0$ will touch the conic $Lx^2+\ldots\ 2L'yz+\ldots = 0$, if $a^2(MN-L'^2)\ldots 2bc\,(M'N'-LL')+\ldots = 0$. This will be satisfied for all interchanges of a, b, c, if $L=M=N=P$ suppose, and $L'=M'=N'=Q$,

that is, if $\quad (a^2+\ldots)(P^2-Q^2)+2\,(bc+\ldots)(Q^2-PQ) = 0$,

that is, $\quad (a^2+\ldots)(P+Q)-2\,(bc+\ldots)\,Q = 0$;

and the conic will be $\quad \frac{x^2+\ldots}{a^2+\ldots} - \frac{2\,(yz+\ldots)}{a^2+\ldots-2bc-\ldots} = 0.$

7241. (By A. McMURCHY, B.A.)—Show that, if m, n, p, q be positive integers, the limiting value, when $x = y = z = a$, of the fraction

$$\frac{x^m(y^n-z^n)+y^m(z^n-x^n)+z^m(x^n-y^n)}{x^p(y^q-z^q)+y^p(z^q-x^q)+z^p(x^q-y^q)}, \text{ is } \frac{mn(m-n)}{pq(p-q)}a^{m+n-p-q}.$$

Solution by W. H. BLYTHE, M.A.; J. O'REGAN; *and others.*

Take $$\frac{x^m(y^n-z^n)+y^m(z^n-x^n)+z^m(x^n-y^n)}{x^2(y-z)+y^2(z-x)+z^2(x-y)},$$

and put $x = a(1+t_1)$, $y = a(1+t_2)$, $z = a(1+t_3)$;

then, substituting, we obtain $\frac{mn(m-n)}{2!}.a^{m+n-2}$ + terms of the form

$$\frac{A.[t_1^b(t_2^c-t_3^c)+t_2^b(t_3^c-t_1^c)+t_3^b(t_1^c-t_2^c)]}{t_1^2(t_2-t_3)+t_2^2(t_3-t_1)+t_3^2(t_1-t_2)},$$

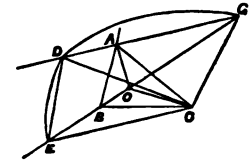
each of the denominators in these fractions evidently cancel into the numerators leaving some powers of t, for $b>2$ and $c>1$; thus we may take as our result $\frac{mn(m-n)}{2!}.a^{m+n-2}$ + powers of t; putting $t_1=t_2=t_3=0$, we get the required result by division.

[If we put $x = a$, $y = a+h$, $z = a+k$, and expand by the Binomial Theorem, we find

$$U = \frac{\frac{mn}{2!}hk\,a^{m+n-3}(m-n)(h-k)+.....}{\frac{pq}{2!}hk\,a^{p+q-3}(p-q)(h-k)+.....} = \text{stated result, in limit.}]$$

7239. (By G. TURRIFF, M.A.)—The angle B and the exterior angle A of a triangle ABC are bisected by straight lines meeting in G. With centre C and radius equal to OG, a circle is drawn cutting GA, GB, or these produced, in D, E. Prove that DE is parallel to AB.

Solution by C. MORGAN, B.A.; A. MUKHOPÂDHYÂY; *and others.*



Bisect $\angle A$ by AO and join OC; then, since

$$AOG = \tfrac{1}{2}A+\tfrac{1}{2}B = 90°-\tfrac{1}{2}C,$$

therefore $$AGO = \tfrac{1}{2}C,$$

therefore a circle goes round AOCG; therefore

$\angle CGO = \tfrac{1}{2}A$, therefore $CGD = \tfrac{1}{2}(A+C)$,

or $$CDG = \tfrac{1}{2}(A+C).$$

Again, $DCE = C$, since it $= 2AGO$ at the circumference, therefore $CDE = 90°-\tfrac{1}{2}C$; therefore

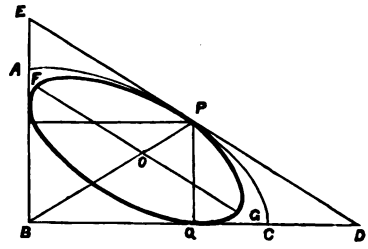
$$EOG = CDE+CDG = 90°+\tfrac{1}{2}A = BAG;$$

therefore AB is parallel to DE.

3267. (By Professor SIVERLY, M.A.)—Determine the axes of the maximum ellipse inscribed in the quadrant of a given ellipse.

Solution by the PROPOSER; Rev. J. L. KITCHIN, M.A.; *and others.*

Let ABC be a quadrant of the given ellipse; from the centre B draw the equal semi-conjugate diameter BP; and at P draw the tangent EPD cutting BA and BC produced in E and D. Let a, b be the semi-axes of the given ellipse; then BP is readily found to be $[\frac{1}{2}(a^2+b^2)]^{\frac{1}{2}}$. The maximum ellipse inscribed in the triangle EBD is known to touch it at the point P; its semi-conjugate diameters are



$OP = \frac{1}{2}BP = \frac{1}{2}[\frac{1}{2}(a^2+b^2)]^{\frac{1}{2}}$, and $OG = \frac{1}{2}BP\sqrt{3} = \frac{1}{2}[\frac{3}{2}(a^2+b^2)]^{\frac{1}{2}}$; its area is $\frac{1}{2}\pi ab\sqrt{3}$; and the axes are

$$\frac{1}{\sqrt{2}}\{[2(a^2+b^2+ab\sqrt{3})]^{\frac{1}{2}}+[2(a^2+b^2-ab\sqrt{3})]^{\frac{1}{2}}\},$$
$$\frac{1}{\sqrt{2}}\{[2(a^2+b^2+ab\sqrt{3})]^{\frac{1}{2}}-[2(a^2+b^2-ab\sqrt{3})]^{\frac{1}{2}}\}.$$

To prove that the inscribed ellipse lies wholly within the quadrant, let O be the origin, OP = B the axis of y, and $OG = b\sqrt{3}$ the axis of x; then the equations of the inscribed and the outer ellipses are

$$x^2+3y^2 = 3B^2,\quad x^2+(y'+2B)^2 = 9B^2 \ldots\ldots\ldots\ldots (1, 2);$$

(2)−(1) gives $$(y'+2B)^2-3y^2 = 6B^2 \ldots\ldots\ldots\ldots (3),$$

or $$-3y^2 = 6B^2-(y'+4By'+4B^2) \ldots\ldots\ldots\ldots (4).$$

Add $3y'^2$ to both members; $3y'^2-3y^2 = 2(y-B^2) \ldots\ldots\ldots\ldots (5)$.

The second member is always positive, hence the first member is always positive, and y' numerically greater than y; which shows that every point of the ellipse ABC lies without the inscribed ellipse, except the point of contact, where $y = y' = B$.

7238. (By W. J. C. SHARP, M.A.)—Prove that the circumscribed circle of a triangle bisects the lines OO_1, OO_2, OO_3, O_2O_3, O_3O_1, O_1O_2, where O, O_1, O_2, O_3 are the centres of the inscribed and circumscribed circles.

Solution by R. W. WHITE, B.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

In this question it is obvious that the circumscribed circle of ABC is the nine-points circle of $O_1O_2O_3$, and that O is the orthocentre of $O_1O_2O_3$; hence the circle round ABC will pass through the points mentioned in the question. [See POWELL's *Geometrical Exercises*, No. 75; Appendix to TODHUNTER's *Euclid*, &c.]

7083. (By Dr. MACFARLANE, F.R.S.E.)—Prove that

$$\frac{1}{[a^2+b^2+2ab\cos(\alpha-\beta)]^{\frac{1}{n}}}\cos n\tan^{-1}\frac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}=\frac{1}{a^n}\left\{\cos n\alpha\right.$$

$$\left.-n\frac{b}{a}\cos[(n+1)\alpha-\beta]+\frac{n(n+1)}{2!}\frac{b^2}{a^2}\cos[(n+2)\alpha-2\beta]-\&c.\right\}$$

$$=\frac{1}{(a\cos\alpha+b\cos\beta)^n}\left\{1-\frac{n(n+1)}{2!}\left(\frac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}\right)^2\right.$$

$$\left.+\frac{n(n+1)(n+2)(n+3)}{4!}\left(\frac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}\right)^4-\&c.\right\}.$$

Solution by the PROPOSER; Rev. J. L. KITCHIN, M.A.; *and others.*

Let a denote a directed quantity of length a and angle α, b a directed quantity of length b and angle β, a and b being in the same plane. Then

$$\frac{1}{(a+b)^n}=\frac{1}{[a^2+b^2+2ab\cos(\alpha-\beta)]^{\frac{1}{n}}}\cdot-n\tan^{-1}\frac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta},$$

where the dot separates the tensor and versor parts,

$$=\frac{1}{[a^2+b^2+2ab\cos(\alpha-\beta)]^{\frac{1}{n}}}$$

$$\times\left\{\cos n\tan^{-1}\frac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}-i\sin n\tan^{-1}\frac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}\right\}.$$

Also $\frac{1}{(a+b)^n}=a^{-n}\left(1+\frac{b}{a}\right)^{-n}=\frac{1}{a^n}-n\frac{b}{a^{n+1}}+\frac{n(n+1)}{2!}\frac{b^2}{a^{n+2}}-\&c.$

$$=\frac{1}{a^n}\left\{\cos n\alpha-n\frac{b}{a}\cos[(n+1)\alpha-\beta]+\frac{n(n+1)}{2!}\frac{b^2}{a^2}\cos[(n+2)\alpha-2\beta]-\&c.\right\}$$

$$-i\frac{1}{a^n}\left\{\sin n\alpha-n\frac{b}{a}\sin[(n+1)\alpha-\beta]+\&c.\right\}.$$

Also $\frac{1}{(a+b)^n}=\frac{1}{[a\cos\alpha+b\cos\beta+i(a\sin\alpha+b\sin\beta)]^n}$

$$=\frac{1}{(a\cos\alpha+b\cos\beta)^n}\left\{1-ni\frac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}\right.$$

$$\left.+\frac{n(n+1)}{2!}i^2\left(\frac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}\right)^2-\&c.\right\}$$

$$=\frac{1}{(a\cos\alpha+b\cos\beta)^n}\left\{1-\frac{n(n+1)}{2!}\left(\frac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}\right)^2+\dots\right\}$$

$$+\frac{-i}{(a\cos\alpha+b\cos\beta)^n}\left\{n\frac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}-\&c.\right\}.$$

Hence the real parts are equal, and the imaginary parts equal.

7279. (By Professor HUDSON, M.A.)—On a given base AB a right-angled spherical triangle ACB is drawn; on CA, CB, CP, CQ are taken each equal to a quadrant; find the loci of P and Q.

Solution by W. H. BLYTHE, M.A.; SARAH MARKS; *and others.*

The angle at C being a right-angle and CP a quadrant, P is the pole of CBQ; therefore BP is always a quadrant and P moves on a great circle of which B is pole. Similarly, Q moves on a great circle of which A is pole.

5650. (By Professor MALET, F.R.S.)—Prove (1) that the locus of the poles of the sides of the triangle of reference with respect to a variable conic which circumscribes it, and which also touches the fixed conic $ax^2+by^2+cz^2=0$, is the curve $a^{\frac{1}{2}}x^{\frac{1}{2}}+b^{\frac{1}{2}}y^{\frac{1}{2}}+c^{\frac{1}{2}}z^{\frac{1}{2}}=0$. Hence (2) show that we can inscribe in this curve a triangle whose sides pass through the vertices of the triangle of reference, any point on the curve being assumed as one vertex. Prove also (3) that, if from any point on one side of the triangle of reference we draw tangents to this curve, the line joining their points of contact passes through the opposite vertex.

Solution by the PROPOSER.

1. The condition that the conic $lyz+mxz+nxy=0$ should touch $ax^2+by^2+cz^2=0$ may be written in the form $a^{\frac{1}{2}}l^{\frac{1}{2}}+b^{\frac{1}{2}}m^{\frac{1}{2}}+c^{\frac{1}{2}}n^{\frac{1}{2}}=0$; and since the coordinates of the poles of the sides of the triangle of reference with respect to the first conic are respectively $(-l, m, n)$, $(l, -m, n)$, $(l, m, -n)$, we have (1).

2. Let x_1, y_1, z_1, be the coordinates of any point on the curve $a^{\frac{1}{2}}x^{\frac{1}{2}}+b^{\frac{1}{2}}y^{\frac{1}{2}}+c^{\frac{1}{2}}z^{\frac{1}{2}}=0$, then the points $(-x_1, y_1, z_1)$ and $(x_1, -y_1, z_1)$ are also points on the curve; if these points be joined, we have a triangle inscribed in the curve whose sides pass through the vertices of the triangle of reference; hence (2).

3. The equations of the tangents to the curve at the points (x_1, y_1, z_1) and $(-x_1, y_1, z_1)$ may be written

$$\left(\frac{a}{x_1}\right)^{\frac{1}{2}}x+\left(\frac{b}{y_1}\right)^{\frac{1}{2}}y+\left(\frac{c}{z_1}\right)^{\frac{1}{2}}z=0, \quad -\left(\frac{a}{x_1}\right)^{\frac{1}{2}}x+\left(\frac{b}{y_1}\right)^{\frac{1}{2}}y+\left(\frac{c}{z_1}\right)^{\frac{1}{2}}z=0,$$

which proves (3); since only two tangents can be drawn to the curve from a point on the line x, which is itself a double cuspidal tangent.

7286. (By W. J. C. SHARP, M.A.)—Prove that perpendiculars drawn through the middle points of the sides of a triangle to the inner bisectors of the angles are the radical axes of the inscribed and the several escribed circles.

Solution by R. KNOWLES, B.A.; W. H. BLYTHE, M.A.; *and others.*

If the inscribed circle touch the side BC at D, and the escribed circle, which touches BC between B and C, touch it at D; then

$$2BD = AB + BC - AC = 2CD,$$

therefore BD = CD, and the mid-point of DD′ is the same as that of BC, or the tangents from this point to the two circles are equal, and this point is on the radical axis of the circles, which is therefore the perpendicular from the middle point of BC on the line joining their centres, *i.e.*, the internal bisector of the angle at A.

Similarly, if a perpendicular be drawn from the middle point of a side of a triangle upon the external bisector of the opposite angle; this is the radical axis of the two escribed circles whose centres lie on their bisector. Hence the radical axes are the bisectors, internal and external, of the angles of the triangle formed by joining the middle point of the sides, and the radical centres are the centres of the inscribed and escribed circles of that triangle.

5712. (By Professor MALET, F.R.S.)—Express in terms of Elliptic Functions the complete integral of the linear equation

$$\frac{d^3y}{dx^3} + \frac{3(1-2x)}{x(1-x)}\frac{d^2y}{dx^2} + \frac{1-7x+7x^2}{x^2(1-x)^2}\frac{dy}{dx} - \frac{1-2x}{2x^2(1-x)^2}y = 0.$$

Solution by the PROPOSER.

Let $x = k^2$, $1-x = k'^2$, then, using JACOBI's notation for Elliptic Integrals, the solution of the equation is found to be

$$y = C_1K^2 + C_2K'^2 + C_3KK',$$

where C_1, C_2, C_3 are arbitrary constants.

7257. (By Professor MALET, F.R.S.)—A cone of the second degree intersects a given sphere in two planes, one fixed, and the other touching in every position another fixed sphere concentric with the first; prove that the locus of the vertex of the cone is a pair of spheres.

Solution by the PROPOSER; Dr. CURTIS; *and others.*

Let the equation of the first fixed sphere be $x^2 + y^2 + z^2 - r^2 = 0$, of the fixed plane $x - a = 0$, and of the variable plane

$$x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0,$$

where, by the conditions of the question, p is constant. The equation of the cone may be written in the form

$$\lambda(x^2+y^2+z^2)-2(x-a)(x\cos\alpha+y\cos\beta+z\cos\gamma-p)=0,$$

where λ is an indeterminate.

To determine the required locus, we have to eliminate λ, $\cos\alpha$, $\cos\beta$, $\cos\gamma$ from the equations

$$\lambda x-(x\cos\alpha+y\cos\beta+z\cos\gamma-p)-(x-a)\cos\alpha=0,$$

$$\lambda y=(x-a)\cos\beta,\quad \lambda z=(x-a)\cos\gamma,$$

$$\lambda r^2-a(x\cos\alpha+y\cos\beta+z\cos\gamma-p)=p(x-a).$$

Now, letting $$x^2+y^2+z^2-2\frac{r^2}{a}x+r^2\equiv S,$$

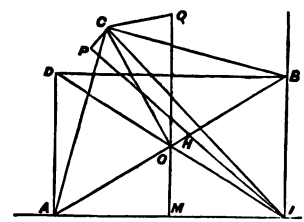
it is found that the eliminant may be put in the form

$$a^2(p^2-a^2)S^2+4ap^2(r^2-a^2)(x-a)S+4p^2r^2(r^2-a^2)(x-a)^2=0,$$

which is manifestly the equation of a pair of spheres.

6994. (By W. Nicolls, B.A.)—A rigid uniform beam, resting on a rough horizontal plane and against an equally rough vertical wall, is prevented from slipping by a string attached to the beam and to the intersection of the planes; the beam lying in a vertical plane perpendicular to the wall, to find the tension and the limiting position of the string.

Solution by the Proposer; Belle Easton; *and others.*



Let $2a$ = length of the beam AB, α = inclination of beam to the horizon, θ = inclination of string, ϵ = angle of friction, w = weight of beam, T = tension of string, O = centre of beam, H = point of attachment of string, Q = intersection of re-actions; then, if DA, DB be perpendiculars to the wall and plane, since $\epsilon=CAD=CBD$, the quadrilateral ABCD is circumscribable by a circle, therefore ACB = ADB = 90°; hence a circle passes through the points A, I, B, C, and D; therefore CO = AO = IO = BO = a.

Again, $CIA=CBA=\alpha+\epsilon$, therefore $CIP=\alpha+\epsilon-\theta$; and, since CI subtends the angle $90°-\epsilon$ at the circumference, it is equal to

$$2a\sin(90°-\epsilon)=2a\cos\epsilon;$$

therefore $$CP=CI\sin CIP=2a\cos\epsilon\sin(\alpha+\epsilon-\theta).$$

Again, $$COA=2\alpha+2\epsilon,\quad\text{and}\quad AOQ=180°-AOM=90°+\alpha,$$

therefore $COQ=90°-\alpha-2\epsilon$; hence $CQ=OC\sin COQ=a\cos(\alpha+2\epsilon)$.

Taking moments round C, we get

$$T \cdot CP = w \cdot CQ, \text{ or } T \cdot 2a \cos\epsilon \sin(\alpha + \epsilon - \theta) = wa \cos(\alpha + 2\epsilon),$$

therefore $$T = \frac{\cos(\alpha + 2\epsilon)}{2\cos\epsilon \sin(\alpha + \epsilon - \theta)}.$$

If $\theta = \alpha + \epsilon$, $T = \infty$, or no tension however great will then keep the beam from slipping. If $\theta > \alpha + \epsilon$, T is negative, which is impossible since the string cannot push. If $\epsilon = 0$, we get the case when both planes are smooth.

7272. (By R. F. Scott, M.A.)—If S and H are the foci of a system of confocal conics, and on SH as diameter a circle is drawn; prove (1) that the locus of the extremities of the diameters conjugate to those drawn to the points where the circle meets the conics of the system, is a lemniscate of Bernoulli; and (2) find what the locus becomes for a circle concentric with the given circle.

Solution by R. Warburton-White, B.A.; J. P. Johnstone; *and others.*

Let Q be the end of the conjugate diameter mentioned, then $CQ^2 = a^2 + b^2 - c^2 = 2b^2 = \rho^2$ suppose. Again, $CQ^2 = \frac{a^2b^2}{a^2 - c^2\cos^2\theta}$, if θ is the vectorial angle of Q. Equating, we get $a^2 = 2a^2 - 2c^2\cos^2\theta$, therefore $c^2(1 + \cos 2\theta) = a^2$, $\therefore a^2 - c^2 = c^2\cos 2\theta$, $\therefore \rho^2 = 2c^2\cos 2\theta$; therefore the locus of Q is a lemniscate of Bernoulli.

6567. (By Professor Nash, M.A.)—In Chasles's *Aperçu Historique*, p. 342, there is given the following method of generating a Cartesian oval, or "ligne aplanétique":—

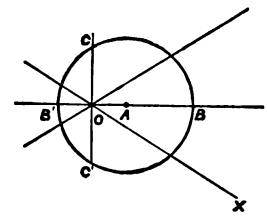
"Etant donnés un cercle et un point fixe, pris arbitrairement dans son plan; si par ce point on mène un rayon vecteur à un point quelconque de la circonférenee du cercle, et une seconde droite, qui fasse avec un certain axe fixe un angle double de celui que fait le rayon vecteur avec cet axe, et qu'on porte sur cette seconde droite, à partir du point fixe, un segment proportionnel au carré du rayon vecteur; l'extrémité de ce segment aura, pour lieu géométrique, une ligne aplanétique formée de deux ovales conjuguées, dont un foyer est au point fixe."

Prove that this method of generation gives only Limaçons, that is, Cartesians in which two single foci coalesce. The fixed point is the single focus of the Limaçon, and the line joining the fixed point to the centre of the circle bisects the angle between the given line and the axis of the Limaçon. The node will be a crunode if the given point lie within the given circle, and an acnode if it lie without the circle. If the point lie

in the circumference of the circle, the node becomes a cusp, and the curve a cardioid.

Solution by the PROPOSER.

Take the fixed point O as origin, the line joining it to the centre A of the circle as axis of x, and let the given line make an angle α with this axis. Let $OA = c$, radius of circle $= a$, and let ρ, ϕ be the polar coordinates of the point P on the locus corresponding to the point Q on the circle. Then in figure



$k \cdot OP = OQ^2$, and $\angle POX = 2\angle QOX = 2\theta$.

The equation of the circle is $(x-c)^2+y^2=a^2$, and the coordinates of Q are $c + a\cos\beta, a\sin\beta$,

therefore $$\frac{\sin(\theta-\alpha)}{a\sin\beta} = \frac{\cos(\theta-\alpha)}{c+a\cos\beta} = \frac{1}{OQ} = \frac{1}{(k\rho)^{\frac{1}{2}}};$$

therefore $$a\sin\beta = (k\rho)^{\frac{1}{2}}\sin(\theta-\alpha) = (k\rho)^{\frac{1}{2}}\sin\tfrac{1}{2}(\phi-\alpha),$$

$$a\cos\beta = (k\rho)^{\frac{1}{2}}\cos(\theta-\alpha) - c = (k\rho)^{\frac{1}{2}}\cos\tfrac{1}{2}(\phi-\alpha) - c;$$

therefore $$a^2 = k\rho - 2c(k\rho)^{\frac{1}{2}}\cos\tfrac{1}{2}(\phi-\alpha) + c^2,$$

therefore $$(k\rho + c^2 - a^2)^2 = 2c^2 k\rho\,[1+\cos(\phi-\alpha)];$$

or, in Cartesian coordinates, putting ρ^2 for x^2+y^2,

$$[k^2\rho^2 + (c^2-a^2)^2 - 2c^2k(x\cos\alpha + y\sin\alpha)]^2 = 4k^2a^4\rho^2.$$

Turning the axes through an angle α in the positive direction, this equation becomes

$$[k^2\rho^2 - 2c^2kx + (c^2-a^2)^2]^2 = 4k^2a^4\rho^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A);$$

therefore the locus of P is a Cartesian whose axis makes an angle α with OA, or OA bisects the angle between the axis and the given line OX.

If the point $(z, 0)$ be a focus, the line $y = i(x-z)$ must touch the curve.

Substituting for y in the equation of the curve, we get

$$4k^2x^2(kz-c^2)^2 + 4kx\left\{(kz-c^2)[(c^2-a^2)^2 - k^2z^2] - za^4k\right\} + [(c^2-a^2)^2 - k^2z^2]^2 + 4a^4k^2z^2 = 0.$$

The condition for equal roots gives $z=0$, or $k^2z^2 - 2kz(c^2-a^2) + (c^2-a^2)^2 = 0$; therefore the origin is a focus, and the other two foci coincide at the point $\left(\frac{c^2-a^2}{k},\ 0\right)$; therefore the curve is a limaçon.

Transferring the origin to the double focus, the equation becomes $[k(x^2+y^2) - 2a^2x]^2 = 4a^2c^2(x^2+y^2)$, or in polar coordinates

$$\rho = \frac{2a^2}{k}\cos\phi \ldots \frac{2ac}{k}.$$

This is the ordinary form of the equation of Pascal's limaçon.

The node will be a crunode, cusp, or acnode, according as c is $<$ $=$ or $> a$, that is, according as the fixed point O lies inside, upon, or outside the given circle. The existence of the node can also be proved directly from the

equation (A); for, if we put $y=0$, we obtain $k^2x^2-2kx\,(c^2\ldots a^2)+(c^2-a^2)^2=0$, and we get two roots equal to $\frac{c^2-a^2}{k}$, the other roots being $\frac{(c\ldots a)^2}{k}$. In the figure, B, B′ are the points on the circle corresponding to the vertices, C, C′ the extremities of a chord through O at right angles to OA, those corresponding to the node.

7237. (By W. J. C. Sharp, M.A.)—If cones be constructed having their vertices at two given points and enveloping a given quadric; prove that (1) their common curve will consist of two conics lying on two planes which pass through the intersection of the polar planes of the given points; and (2) if these points lie on a quadric concentric, similar, and similarly situated to the given one, one of the planes will pass through the centre, and the other will be the polar plane of the middle point of the line joining the given points.

Solution by Dr. Curtis; Christine Ladd, B.A.; *and others.*

If (h, k, l), (h', k', l') be the points, and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ the quadric, the equations to the intersection of the enveloping cones will be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=\left(\frac{hx}{a^2}+\frac{ky}{b^2}+\frac{lz}{c^2}-1\right)^2+\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}+\frac{l^2}{c^2}-1\right)$$
$$=\left(\frac{h'x}{a^2}+\frac{k'y}{b^2}+\frac{l'z}{c^2}-1\right)^2+\left(\frac{h'^2}{a^2}+\frac{k'^2}{b^2}+\frac{l'^2}{c^2}-1\right),$$

and will lie on the planes

$$\left(\frac{hx}{a^2}+\frac{ky}{b^2}+\frac{lz}{c^2}-1\right)\left(\frac{h'^2}{a^2}+\frac{k'^2}{b^2}+\frac{l'^2}{c^2}-1\right)^{\frac{1}{2}}$$
$$=\pm\left(\frac{h'x}{a^2}+\frac{k'y}{b^2}+\frac{l'z}{c^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}+\frac{l^2}{c^2}-1\right)^{\frac{1}{2}}\ldots\ldots(1),$$

which pass through the intersection of the polar planes of (h, k, l) and (h', k', l'), and represent the polar planes of the two points which cut internally and externally, in the ratio

$$\left(\frac{h^2}{a^2}+\frac{k}{b^2}+\frac{l^2}{c^2}-1\right)^{\frac{1}{2}}:\left(\frac{h'^2}{a^2}+\frac{k'^2}{b^2}+\frac{l'^2}{c^2}-1\right)^{\frac{1}{2}},$$

the line joining the two given points. If the points (h, k, l), (h', k', l') lie on a quadric concentric, similar, and similarly situated to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0,$$

the two quantities under the radical in (1) are equal, and the plane obtained by taking the upper sign passes through the centre, while the other

$$\frac{(h+h')\,x}{a^2}+\frac{(k+k')\,y}{b^2}+\frac{(l+l')\,z}{c^2}=2$$

is the polar plane of the middle point of the line joining (h, k, l) and (h', k', l').

7219. (By Professor SYLVESTER, F.R.S.) — Prove that a series of continually increasing integers, no one of which is a sum of positive multiples of all or any of those that precede it, cannot be continued *ad infinitum*.

Solution by the PROPOSER.

1. Suppose that one of the integers written down is a prime number p. Then, as is well known, if n be any other number of the series, $xp + yn$ may be made equal with positive values of x, y to any number not less than pn. Hence, the series in this case comes to an end as soon as pn is reached or surpassed.

2. Suppose the theorem to be true when one of the integers written down contains only i prime factors. Then I say it will be true if any one of the same contains $i + 1$ factors. For, suppose one of the numbers P to be $p^\alpha . q^\beta . r^\gamma \ldots$, where $\alpha + \beta + \gamma \ldots = i + 1$.

It is obvious that only a finite number of terms of the series can contain either p, or q, or $r \ldots$. For, if $p^\alpha . q^\beta . r^\gamma \ldots$ and pm, pm', pm'', ... could be continued indefinitely, the same would be true of

$$p^{\alpha-1} . q^\beta . r^\gamma \ldots, \quad m, m', m'', \ldots,$$

contrary to the hypothesis. Let, then, n be any integer of the series not divisible by p, or q, or $r \ldots$, *i.e.*, prime to P; then $x\mathrm{P} + yn$ can be made equal to any number not less than $\mathrm{P}n$; and consequently what remains of the series, after the previous different limited series are struck out, cannot be continued *ad infinitum*; and the theorem is true for the case where one of the numbers contains $i + 1$ factors, if it is true when one of the numbers contains only i factors; and, being true when i is unity, is true universally.

7166. (By Professor TOWNSEND, F.R.S.) — The central axis of a system of three forces in a common space being supposed to pass through the centre of the quadric determined by their three lines of action in the space; show, on elementary principles, that it must coincide with a principle axis of the surface.

Solution by the PROPOSER.

For since, on transference of the three forces to the centre of the quadric, the plane of their resultant moment is always conjugate to the direction of their resultant force with respect to the surface (see Quest. 6769); therefore, &c.

5250. (By the late Dr. BOOTH, F.R.S.) — Through a given point within an angle draw (1) the minimum line between the sides of the given angle; and (2) show that, when the angle is a right angle, the minimum line drawn through the given point is a tangent to the quadrantal hypocycloid passing through the same point.

Solution by J. HAMMOND, M.A.

Taking the sides of the given angle α as axes, the equation to any line through a given point within it (a, b) is $y-b = m(x-a)$..... (1), and, since the intercepts of this line on the axes are $a-\frac{b}{m}$ and $b-am$, the square of its length is $\left(\frac{am-b}{m}\right)^2(1+m^2+2m\cos\alpha) = \text{min.}$ .....(2). Differentiating (2) with respect to m, and removing the superfluous factor $2\left(\frac{am-b}{m^3}\right)$, we have $am^3+am^2\cos\alpha+bm\cos\alpha+b=0$ ..... (3). Now, both a and b being positive, (3) has always one real negative root, and the value of this root substituted in (1) gives the equation to the minimum line, as is obvious from geometrical considerations.

When $\cos\alpha = 0$, we have to substitute $m = -\left(\frac{b}{a}\right)^{\frac{1}{2}}$ in (1), which then becomes $(x-a)a^{-\frac{1}{2}}+(y-b)b^{-\frac{1}{2}}=0$, which touches $x^{\frac{1}{2}}+y^{\frac{1}{2}} = a^{\frac{1}{2}}+b^{\frac{1}{2}}$ at the point (a, b).

7338. (By S. TEBAY, B.A.)—The parallel sides of a trapezoid are $m^2+2mn-n^2$ and $n^2+2mn-m^2$; show that it is divided into two equal trapezoids by the line m^2+n^2.

Solution by W. W. TAYLOR, M.A.; R. KNOWLES, B.A.; *and others.*

The distances between the extreme lines and the mean are proportional to $2(mn-n^2)$ and $2m^2-2mn$; hence the areas are proportional to

$$2(m-n)n(2m^2+2mn) \text{ and } 2m(m-n)(2n^2+2mn),$$

and these are each equal to $4mn(m^2-n^2)$; hence the two areas are equal.

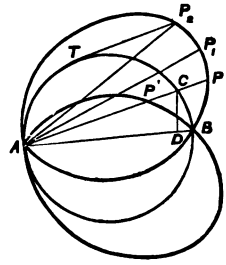
7034. (By the EDITOR.)—If ACB be a circle on AB as diameter, and P, P two points on the vector AC such that CP = CP′ = distance (CD) of C from AB; show that (1) the locus of P is a heart-shaped curve circumscribing the circle, and of P′ a continuous loop inscribed therein; (2) the areas of these loci, in parts of the area of the circle, are $\frac{5}{4} \pm \frac{8}{3\pi}$;

(3) the maximum vector $AP_1 (= \frac{3}{4}\sqrt{3} \cdot AB)$ makes an angle $P_1AB = 30°$ with AB; (4) the maximum tangent P_2T from P to the circle makes an angle $P_2AB = 39° 49' 15''$ with AB; and (5) the revolution of AP_2 round AB generates the cone of greatest entire surface that can be cut from the sphere on AB as diameter.

Solution by W. H. Blythe, M.A.; Prof. Matz, M.A.; *and others.*

1. Taking A as pole, and AB as polar axis, both curves are included in the equation

$$r = 2a \cos\theta (1 + \sin\theta).$$



When $\theta = 0$, $r = 2a$; as θ increases, since

$$dr : d\theta = 2a (-\sin\theta + 2\cos 2\theta),$$

we find that, so long as $2\cos 2\theta > \sin\theta$, r increases; when $2\cos 2\theta = \sin\theta$, r has its greatest value; after this, $dr : d\theta$ being negative, r decreases as θ increases. When $\theta = \frac{1}{2}\pi$, $r = 0$; as θ increases from $\frac{1}{2}\pi$ to π, r is negative, and the value of r geometrically is the same for θ as $\pi - \theta$; thus we obtain a curve below AB similar to that traced above. When $\theta = \pi$, $r = -2a$, and we again arrive at B. As θ increases r diminishes, and is negative (that is, algebraically increases), and $dr : d\theta$ being positive from $\theta = \pi$ to $\frac{3}{2}\pi$ we have a continuous curve within the circle, when $\theta = \frac{3}{2}\pi$, $r = 0$. From $\theta = \frac{3}{2}\pi$ to 2π, r is positive, and equal (with the signs changed and order reversed) to the values of r found from π to $\frac{3}{2}\pi$; thus we have a similar continuous curve (within the circle) to that traced above, but on the other side of AB.

2. For the area of the outer curve, integrate

$$\int_0^{\frac{1}{2}\pi} \tfrac{1}{2} [2a \cos\theta (1 + \sin\theta)]^2 . d\theta,$$

taken twice over, which reduces to

$$4a^2 \int_0^{\frac{1}{2}\pi} (\cos^2\theta + 2\cos^2\theta \sin\theta + \cos^2\theta \sin^2\theta)\, d\theta = 4a^2 (\tfrac{1}{4}\pi + \tfrac{2}{3} + \tfrac{1}{16}\pi) \ldots\ldots (1).$$

For the second (or continuous curve within) integrate the same expression twice from π to $\frac{3}{2}\pi$, or (which is the same thing) substitute $\pi + \theta$ for θ and integrate from 0 to $\frac{1}{2}\pi$; thus we obtain

$$4a^2 \int_0^{\frac{1}{2}\pi} (\cos^2\theta - 2\cos^2\theta \sin\theta + \cos^2\theta \sin^2\theta)\, d\theta = 4a^2 (\tfrac{1}{4}\pi - \tfrac{2}{3} + \tfrac{1}{16}\pi) \ldots\ldots (2).$$

Results (1) and (2) are both included in $\pi a^2 \left\{ \frac{5}{4} \pm \frac{8}{3\pi} \right\}$.

3. It has been shown above that the maximum value for r is given by $2\cos 2\theta = \sin\theta$ when $\theta < \frac{1}{2}\pi$, the solution to which equation is evidently $\theta = 30°$. Taking $r = 2a \cos\theta (1 + \sin\theta)$, we obtain

$$r = 2a \cdot \tfrac{1}{2}\sqrt{2} \cdot \tfrac{3}{2} = \tfrac{3}{4} \cdot \sqrt{3} \cdot (2a).$$

4. The length of a tangent from (r_1, θ_1) to the circle is known to be $(r_1^2 - 2ar_1 \cos\theta_1)^{\frac{1}{2}}$. If we take (r_1, θ_1) on the curve, the question reduces to making $\cos^2\theta_1 (1 + \sin\theta_1) \sin\theta_1$ a maximum; thus, taking $\sin\theta = x$ as the variable, we obtain $t^2 = 4a^2 (x + x^2 - x^3 - x^4)$, where t is the length of

the tangent. The maximum value of t is found from $1+2x-3x^2-4x^3=0$; and, dividing by $1+x$ (evidently not a root), we obtain $4x^2=x+1$, $x=\frac{1}{8}(1\pm\sqrt{17})$, giving as the evident solution

$\sin\theta_1=\frac{1}{8}(1+\sqrt{17})=\cdot640393$, &c., whence $\theta_1=P_2AB=39°\,49'\,15''$.

5. A cone cut from the sphere on AB as diameter, by the revolution of a line AP where $PAB=\theta$ round AB, has its entire surface given by

$$\text{curved surface}=2a\cos\theta\,.\,\pi a\sin 2\theta=4a^2\pi\sin\theta(1-\sin^2\theta),$$

$$\text{area of base}=(\pi a\sin 2\theta)^2=4a^2\pi\sin^2\theta(1-\sin^2\theta),$$

therefore entire surface $=4a^2\pi(\sin\theta+\sin^2\theta-\sin^3\theta-\sin^4\theta)=\pi t^2$;

hence the surface is a maximum when t is a maximum, and the position of AP coincides with AP_2.

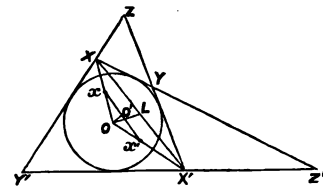
7315. (By W. S. M'Cay, M.A.)—If XX', YY', ZZ' be the diagonals of a quadrilateral circumscribed to a circle, L, M, N their mid-points, and O the centre of the circle; prove that

$$\frac{OX\,.\,OX'}{OL}=\frac{OY\,.\,OY'}{OM}=\frac{OZ\,.\,OZ'}{ON}$$

Solution by J. Young, M.A.; Prof. Nash, M.A.; *and others.*

Take x, x' the inverse points to X, X'; let O' be the middle point of xx'; and join Oo'.

The triangles OXX', $Ox'x$ being similar, we have $\frac{OX}{OL}=\frac{OX'}{OO'}$.



Hence it follows that

$$\frac{OX\,.\,OX'}{OL}=\frac{Ox'\,.\,OX'}{OO'}=\frac{R^2}{OO'}.$$

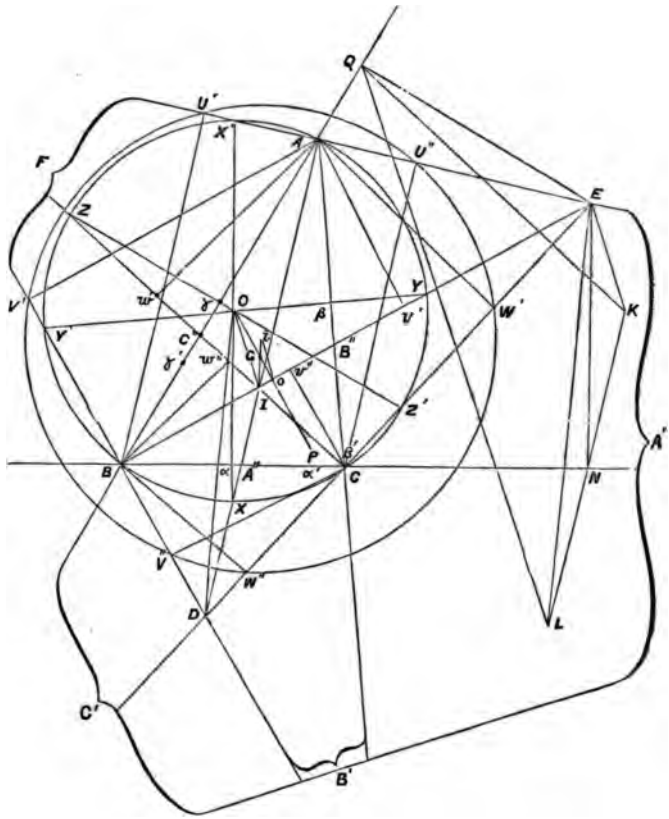
But O' is also the middle point of yy', and z, z' the inverses of YY', ZZ'; for, since x, x', y, y', z, z' are the mid-points of the sides and diagonals of the inscribed quadrilateral, xx' and yy' are diagonals of a parallelogram, as are also xx' and zz', and O' is their common mid-point, so that each of the given fractions is equal to $R^2\div Oo'$.

7185. (By Morgan Jenkins, M.A.)—Show, as a special case of Gergonne's general construction for describing a circle to touch three given circles (Salmon's *Conics*, Ch. ix., Art. 127), that the nine-points-circle of a triangle touches the three escribed circles of that triangle; also, in like manner, that it touches two of the escribed circles and the inscribed circle; show also that the three circles, each of which touches two of the escribed

circles externally and one internally, pass through the same point, viz., the centre of the circle inscribed in the triangle obtained by joining the middle points of the sides of the first triangle.

Solution by the PROPOSER.

Let ABC be the triangle; α, β, γ the middle points of its sides; α', β', γ' the feet of the perpendiculars from the corners on the sides; D, E, F the centres of the escribed circles, AD, BE, and CF meeting in I, the centre of the inscribed circle; let BC, CA, and AB meet EF, FD, DE in A′, B′, C′ respectively, and AD, BE, CF in A″, B″, C″ respectively; let



BU′, CU″ be perpendiculars from B, C on EF; in like manner AV′, CV″ on FD, and AW′, BW″ on DE; Av', Cv'' on IE; Aw', Bw'' on CI; O the centre of the circle circumscribing ABC; P the orthocentre; υ the middle

point of OP or centre of the nine-points-circle; G the centroid of the triangle ABC, so that GP = 2GO and GO = 2Go; measure Gi = ½GI and in the opposite direction; then io is parallel to OI and = ½OI. Draw EN, EQ perpendicular to BC, BA produced; and draw NK, QK perpendicular to FE, DE, respectively meeting in K; let Oα, Oβ, and Oγ meet the circle ABC in X, Y, Z respectively, on opposite sides to A, B, C respectively; and in X′, Y′, Z′ on the same side. Then, for brevity, denoting circles by their centres, A′, B′, C′ are the external centres of similitude of the circles D, E, and F, and A′ B′ C′ an axis of similitude.

Since NK, QK are perpendicular to A′E, C′E, and NA′, QC′ are tangents to the circle E; therefore K is the pole of A′B′C′ with regard to the circle E; and therefore EK is perpendicular to A′B′C′; also

$$\alpha N = \alpha C + CN = \tfrac{1}{2}a + s - a = s - \tfrac{1}{2}a;$$

similarly for point of contact with BC of circle F, therefore α is on the radical axis of circles E and F; also, because G is the internal centre of similitude or centre of negative inversion of the circles O and o, therefore $i\alpha$ is parallel to and = ½IA, therefore $i\alpha$ is perpendicular to EF, and is therefore the radical axis of the circles E and F; similarly for $i\beta$ and $i\gamma$; and therefore i is the radical centre of the circles D, E, and F.

Again,

$$\frac{EU'}{EB} = \cos\tfrac{1}{2}C = \frac{EN}{EC}, \text{ and } \frac{EU''}{EC} = \cos(\tfrac{1}{2}A + \tfrac{1}{2}C) = \sin\tfrac{1}{2}B = \frac{EQ}{EB};$$

hence EU′ . EU″ = EN . EQ = r_2^2; similarly EW′ . EW″ = r_2^2. Therefore U′, U″, W′, W″ lie on a circle of which i will be the centre, because $i\alpha$, $i\gamma$ bisect U′U″ and W′W″ respectively at right angles. Therefore iU′ = iU″ = iW′ = iW″; similarly iW′ = iW″ = iV′ = iV″, therefore the six points U′, U″, V′, V″, W′, W″ lie on a single circle which from the relation EU′ . EU″ = r_2^2, and, from similar relations, cuts the circles D, E, and F orthogonally. Since OX, OZ are parallel to EN, EQ respectively, and in the same ratio, viz., 1 : 1; also XI, ZI, being perpendicular to EF, DE respectively, are parallel to NK, QK respectively, therefore the remaining line OI is parallel to the remaining line EK, that is, OI is perpendicular to A′B′C′; and the triangle OIX is similar and similarly situated to triangle EKN; therefore

$$\frac{OI}{EK} = \frac{OX}{EN} = \frac{R}{r_2}.$$

Now, from the negative inversion with regard to G, oi = ½OI and is parallel to OI, therefore io is perpendicular to A′B′C′; and $\frac{io}{EK} = \frac{\frac{1}{2}R}{r_2}$.

Again, from Euc. III., 36, Cor., A′A . A′X′ = A′B . A′C; and

$$\frac{A'a'}{A'A} = \frac{A'a}{A'X'} = \frac{A'U'}{A'B} = \frac{A'U''}{A'C} = \cos AA'a',$$

therefore A′a′ . A′a = A′U′ . A′U″, therefore A′ is on the radical axis of the circles i and o; in like manner for B′ and C′; therefore A′B′C′ is the radical axis of the same two circles. Also A′B′C′ is an axis of similitude, and i the radical centre of the three circles D, E, F; io is perpendicular to A′B′C′, and $\frac{io}{EK} = \frac{\frac{1}{2}R}{r_2}$; therefore o is the centre and ½R the radius of a circle touching each of the circles D, E, and F.

In like manner, a single circle will pass through the six points U′, U″, v', v'', w', w'', cutting the three circles I, E, and F orthogonally, and

having for centre the point d, which is the corresponding point to D with regard to G as a centre of negative inversion; that is, $Gd = \frac{1}{4}GD$, and in the opposite direction. Then d is the radical centre of the three circles E, F, and I, da, $d\beta$, and $d\gamma$ being the radical axes of these circles taken in pairs; and A′B″C″ is an axis of similitude of these three circles. Draw QL perpendicular to EC″ and NL perpendicular to EA′, so that L is the pole of A′B″C″ with regard to the circle E, and therefore EL perpendicular to A′B″C″. Now, since the points C, I, C″, F form a harmonic range, and also ∞, F, Z, I; and since EC is perpendicular to D∞ or FC, EI perpendicular to DF, and EF perpendicular to DI; therefore also EC″ will be perpendicular to DZ. Therefore QL is parallel to DZ; but EQ, EN are parallel and proportional to OZ and OX; and NL is parallel to DX, therefore EL is parallel to OD; but do is parallel to OD, therefore do is parallel to EL, and therefore perpendicular to A′B″C″; and, as before, A′B″C″ can be proved to be the radical axis of the circles d and o; and $\frac{do}{EL} = \frac{\frac{1}{2}R}{r_2}$; therefore the circle o touches the three circles E, F, and I.

Again, the straight line A′CB may be regarded as a circle of infinite radius touching the circles E and F internally, and the circle D externally; therefore i, the radical centre of the three circles D, E, and F, will be a centre of similitude of this straight circle and the associated touching circle, viz., that which touches the circles E and F externally and the circle D internally; but the centres of similitude of a straight line and a circle lie on the circle; therefore the last-mentioned touching circle passes through i. It may be observed that this touching circle must also pass through the two points of intersection of A′BC with the circle i; for A′BC is the axis of similitude from which the points of contact of the straight circle A′BC and the associated touching circle may be derived. Hence A′BC must be a common radical axis to these two touching circles and the orthogonal circle i.

7313. (By C. Leudesdorf, M.A.)—Prove that the equations

$$\frac{ad-bc}{2(ac-b^2)} = \frac{ae+2bd-3c^2}{3(ad-bc)} = \frac{3(be-cd)}{ae+2bd-3c^2} = \frac{2(ce-d^2)}{be-cd}$$

are equivalent to $ae+3c^2-4bd=0$, $ace+2bcd-ad^2-eb^2-c^3=0$.

Solution by J. Hammond, M.A.; R. Knowles, B.A.; *and others.*

The Hessian of the quartic is (see Salmon's *Higher Algebra*, ed. 3, p. 190)

$$H = (ac-b^2)x^4 + 2(ad-bc)x^3y + (ae+2bd-3c^2)x^2y^2 + 2(be-cd)xy^3 + (ce-d^2)y^4.$$

Hence, writing λ for the common value of the fractions, the conditions of the question are equivalent to $H = (ac-b^2)(x+\lambda y)^4$.

The corresponding peculiarity of the quartic is that three of its roots

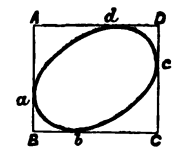
are equal, in which case the quartic may by linear transformation be reduced to the form x^3y; and generally, when a quantic has all its roots but one equal, its Hessian is a perfect power. For, if $U = x^{n-1}y$, the Hessian is $$\frac{d^2U}{dx^2}\cdot\frac{d^2U}{dy^2}-\left(\frac{d^2U}{dx\,dy}\right)^2 = -(n-1)^2x^{2n-4}.$$

Now, since $(a, b, c, d, e\,\mathcal{X}x, y)^4$ has three equal roots (each equal to $-\lambda$), the required conditions can be found by eliminating between the quartic itself and its first and second derived functions $(a, b, c, d\,\mathcal{X}x, y)^3$ and $(a, b, c\,\mathcal{X}x, y)^2$; and the conditions are—(1) discriminant of quartic = 0, or $$(ae-4bd+3c^2)^3-27(ace-ad^2-eb^2+2bcd-c^3)^2 = 0;$$
(2) Discriminant of cubic = 0, or
$$(ac-b^2)(ae-4bd+3c^2)-a(ace-ad^2-eb^2+2bcd-c^3) = 0,$$
which are obviously equivalent to the conditions of the question; viz., both invariants of the quartic vanish.

[Substituting, in the first series of equations, $ae = 4bd-3c^2$, each equation reduces to $6abcd-a^2d^2-4b^3d+3b^2c^2-4ac^3 = 0$, and this, with the same substitution, is identically the same as
$$ace+2bcd-ad^2-eb^2-c^3 = 0.]$$

7225. (By Professor CAVALLIN, M.A.)—If a closed convex curve glides always in contact with two fixed straight lines, and rotates until returning to its original position; prove that the mean value of the arc comprised between the points of contact is the same for all curves that have the same perimeter.

Solution by G. F. WALKER, M.A.



Let ABCD be a rectangle circumscribed to the curve touching at *abcd*.

Then *ab*, *bc*, *cd*, *da* are four possible values of the intercepted arc, and their mean is ¼ of the perimeter, and this is the same however the rectangle be drawn; therefore, &c.

The same method will do if the curve slip between two lines inclined at an angle $\frac{2\pi}{n}$, and the mean value of the intercepted arc will then be one-n^{th} of the perimeter. If the angle be $\frac{2m\pi}{n}$ we get a polygon surrounding the curve m times, and the method applies, the mean being $\frac{m}{n}$ of the perimeter. If the angle be not commensurable with 2π, we may still prove the mean to be $\frac{a}{2\pi}$ of the perimeter.

[In each of the latter cases the *external* angle between the tangents is taken as their *inclination*.]

7311. (By J. J. WALKER, M.A.)—If a point P describes a circumference of radius a with a velocity varying as the square of the distance (r) from any fixed point in its plane, show that the acceleration makes with the circumference at P an angle ψ determined by

$$\cot\psi = -2a\cos\phi \div r,$$

ϕ being the angle which r makes with the circumference at P; and that the acceleration varies as $r^3\cos\phi \div \cos\psi$.

Solution by A. H. CURTIS, LL.D.; J. O'REGAN; *and others.*

Equating tangential and normal components to rate of alteration in velocity and centrifugal force, since $V = \mu r^2$, we get

$$F\sin\psi = \frac{V^2}{a} = \frac{\mu^2 r^4}{a} \quad -F\cos\psi = \frac{dV}{dt} = \frac{2\mu\, r\, dr}{dt} = 2\mu r\cos\phi\,\frac{ds}{dt}$$

$$= 2\mu r V\cos\phi = 2\mu^2 r^3\cos\phi, \text{ or } -F\cos\psi = 2\mu^2 r^3\cos\phi;$$

dividing (2) by (1), $-\cot\psi = \dfrac{2a\cos\phi}{r}$, therefore $\cos\phi = -\dfrac{r}{2a}\cot\psi$.

Squaring ($F\sin\psi$) and ($F\cos\psi$), and adding,

$$F^2 = (\mu^2 r^3)^2\left\{\frac{r^2}{a^2} + 4\cos^2\phi\right\}$$

$$= (\mu^2 r^3)^2\left\{4\cos^2\phi\tan^2\psi + 4\cos^2\phi\right\}$$

$$= (2\mu^2 r^3\cos\phi)^2\left\{\tan^2\psi + 1\right\} = (2\mu^2 r^3\cos\phi)^2\sec^2\psi;$$

therefore $F = 2\mu^2 r^3\cos\phi\sec\psi = 2\mu^2 r^3\cos\phi \div \cos\psi$.

7229. (By Prof. MARTIN, M.A., Ph.D.)—Show that the value of the integral

$$\int_0 [(1+e^2)\,E(e,1) - (1-e^2)\,F(e,1)]\,de,$$

where $F(e,1)$ and $E(e,1)$ denote complete elliptic functions of the first and second kinds, is ·72445.

Solution by W. H. BLYTHE, M.A.; SARAH MARKS; *and others.*

By expansion by the Binomial Theorem and integration, it may easily be shown that the sum required

$$= \pi\left[\tfrac{1}{3} - \Sigma_1^\infty a_n^2\left\{\frac{4n+1}{(2n-1)(2n+1)(2n+3)}\right\}\right], \text{ where } a_n = \frac{2n-1}{2n}\cdot a_{n-1}.$$

It is to be noticed that

$$\Sigma_{n+1}^\infty a_m^2\,\frac{4m+1}{(2m-1)(2m+1)(2m+3)} < a_{n+1}^2\,\Sigma_{n+1}^\infty\left\{\frac{4m+1}{(2m-1)(2m+1)(2m+3)}\right\}$$

$$= a_{n+1}^2\left\{\frac{8n-1}{4\,(2n+1)(2n-1)}\right\}.$$

Hence we may state that the finite value required lies between

$$\pi\left\{\tfrac{1}{3}-\Sigma_0^n \frac{(4m+1)\,a_m^2}{(2m-1)(2m+1)(2m+3)}\right\}$$

and $$\pi\left\{\tfrac{1}{3}-\Sigma_0^n \frac{(4m+1)\,a_m^2}{(2m-1)(2m+1)(2m+3)} - \frac{(8n-1)\,a_{n+1}^2}{4(4n^2-1)}\right\}.$$

Calculating the first sixteen terms by the aid of logarithms, we find the integral falls between $\pi\{\cdot23088\}$ and $\pi\{\cdot23050\}$, (the latter being probably nearer the correct value): taking ·2306, we arrive at ·72445 as the final result.

7266. (By Dr. MACFARLANE, F.R.S.E.)—Find the number of different possible states of the poll, under the cumulative system of voting, when there are p candidates, q seats, and r electors.

Solution by the PROPOSER.

The number of different states of the poll when there are r electors, each having q cumulative votes, is the same as when there is but one elector, having rq cumulative votes. Hence (see solution to previous problem) the number of different states of the poll is $\frac{p+qr\,!}{p\,!\;qr\,!}$.

In the particular election there referred to, this number was, in round numbers, 47119×10^{116}.

5585. (By L. H. ROSENTHAL, B.A.)—If we have given

$$(p^2-4r)(q^2-4t) = (pq-2s)^2,$$

(1) solve completely the sextic equation,

$$x^6+px^4+qx^3+rx^2+sx+t=0;$$

and give (2) the corresponding condition of solution, when the second term is *not* missing.

Solution by J. HAMMOND, M.A.

1. If for an instant we write $x^3 = y$ the sextic becomes

$$y^2+(px+q)\,y+(rx^2+sx+t) = 0,$$

the solution of this regarded as a quadratic in y is

$$2y+px+q = \pm[(px+q)^2-4(rx^2+sx+t)]^{\frac{1}{2}}.$$

And if $p^2-4r = \alpha^2$, $pq-2s = \alpha\beta$, $q^2-4t = \beta^2$, we have, on replacing y by x^3, the two cubics $2x^3+px+q = \pm(\alpha x+\beta)$ which are immediately soluble.

2. The corresponding condition for the sextic

$$x^6+6bx^5+15cx^4+20dx^3+15ex^2+6fx+g=0,$$

may be found by writing $x-b$ in place of x, when the equation becomes

$$\begin{aligned}
&x^6-6bx^5+15b^2x^4-20b^3x^3+15b^4x^2-\ 6b^5x+b^6\\
&+6bx^5-30b^2x^4+60b^3x^3-60b^4x^2+30b^5x-6b^6\\
&+15cx^4-60bcx^3+90b^2cx^2-60b^3cx+15b^4c\\
&+20dx^3-60bdx^2+60b^2dx-20b^3d\\
&+15ex^2-30bex+15b^2e\\
&+6fx-6bf\\
&+g=0,
\end{aligned}$$

and the values $p=15(c-b^2)$, $q=20(d-3bc+2b^3)$, $r=15(e-4bd+6b^2c-3b^4)$, $s=6(f-5be+10b^2d-10b^3c+4b^5)$, $t=g-6bf+15b^2e-20b^3d+15b^4c-5b^6$ must be substituted in the condition $(p^2-4r)(q^2-4t)=(pq-2s)^2$.

7309. (By W. J. C. Sharp, M.A. Suggested by Question 7249.)—If S_r denote the value of the sum of the r^{th} homogeneous products of any quantities, and s_r the sum of the r^{th} powers of the same quantities; show that

$$S_1=s_1,\quad 2S_2=S_1s_1+s_2,\quad 3S_3=S_2s_1+S_1s_2+s_3,$$

and

$$rS_r=S_{r-1}s_1+S_{r-2}s_2+S_{r-3}s_3+\ldots+s_r.$$

Solution by J. Hammond, M.A.; Professor Matz, M.A.; *and others.*

Let the quantities be $a_1, a_2, a_3, \ldots a_n$; then it is easily seen that

$$\begin{aligned}
&\frac{1}{(1-a_1x)(1-a_2x)(1-a_3x)\ldots(1-a_nx)}\\
&=1+xS_1+x^2S_2+x^3S_3+\ldots+x^rS_r+\ldots=P \text{ suppose}\ldots\ldots(1),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{1-a_1x}+\frac{1}{1-a_2x}+\frac{1}{1-a_3x}+\ldots+\frac{1}{1-a_nx}\\
&=n+xs_1+x^2s_2+x^3s_3+\ldots+x^rs_r+\ldots=Q \text{ suppose}\ldots\ldots(2).
\end{aligned}$$

Now $$x\frac{d}{dx}(\log P)=\frac{x}{P}\frac{dP}{dx}=\frac{a_1x}{1-a_1x}+\frac{a_2x}{1-a_2x}+\ldots=Q-n.$$

So that the coefficient of x^r in $x\dfrac{dP}{dx}$ = the coefficient of x^r in $P(Q-n)$,

i.e., $$rS_r=\text{co. } x^r \text{ in } (1+xS_1+x^2S_2+\ldots)(xs_1+x^2s_2+\ldots).$$

Or $$rS_r=S_{r-1}s_1+S_{r-2}s_2+S_{r-3}s_3+\ldots+S_1s_{r-1}+s_r \quad\ldots\ldots\ldots\ldots (3).$$

It would not be difficult to verify Algebraically the particular cases

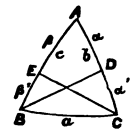
$$S_1=s_1,\quad 2S_2=S_1s+s_2,\quad 3S_3=S_2s_1+S_1s_2+s_3.$$

7267. (By C. Morgan, B.A.)—If arcs of great circles be drawn from the angles A and B of a spherical triangle perpendicular to the opposite sides; prove that, if $\tan\alpha\tan\alpha' = \tan\beta\tan\beta'$, where $\alpha, \alpha', \beta, \beta'$ are the segments into which the sides are divided, the triangle must be either isosceles or right-angled.

Solution by Professor Roy, M.A.; Belle Easton; *and others.*

Since the angle at D is a right angle, we have

$$\tan\alpha' = \frac{\cos C}{\cot a}; \quad \text{similarly } \tan\alpha' = \frac{\cos A}{\cot c}.$$



$$\therefore \quad \tan\alpha\,.\,\tan\alpha' = \frac{\cos A\,.\,\cos C}{\cot a\,.\,\cot c} = \tan\beta\,.\,\tan\beta'$$

$$= \frac{\cos A\,.\,\cos B}{\cot a\,.\,\cot b}.$$

Hence either $\cos A = 0$, which gives $A = 90°$, or

$$\cot b\,.\,\cos C = \cos B\,.\,\cot c.$$

$$\cos a\,.\,\sin b = \sin a\,.\,\cos b\,.\,\cos C + \sin c\,.\,\cos A,$$

$$\cos a\,.\,\sin c = \sin a\,.\,\cos c\,.\,\cos B + \sin b\,.\,\cos A,$$

$$\therefore \quad \cot a = \cos C\,.\,\cot b + \frac{\sin C\,.\,\cos A}{\sin a\,.\,\sin b} = \cos B\,.\,\cot c + \frac{\sin b\,.\,\cos A}{\sin a\,.\,\sin c},$$

therefore $\sin^2 C = \sin^2 b$, therefore $\sin C = \sin B$, therefore $C = B$. Therefore the triangle is either isosceles or right-angled.

7235. (By C. Leudesdorf, M.A.)—Each of three circles (radii ρ_1, ρ_2, ρ_3) touches two sides of a plane triangle ABC and the nine-point circle, having external contact with the latter; if R, r be the radii of the circumscribed and inscribed circles of ABC, and Δ its area, prove that

$$(r-\rho_1)(r-\rho_2)(r-\rho_3) = 2R^2r^5\Delta^{-2}.$$

Solution by W. S. M'Cay, M.A.

Since the nine-point circle touches the escribed circles, any one of the circles in question is the inverse of the corresponding escribed circle with regard to a circle cutting the nine-point circle orthogonally, and having its centre at the corresponding vertex of the triangle; hence, immediately, the radii of the three circles are

$$\rho_1 = r\frac{\cos A}{1+\cos A}, \quad \rho_2 = r\frac{\cos B}{1+\cos B}, \quad \rho_3 = r\frac{\cos C}{1+\cos C},$$

from which the relation in question follows easily. If the circles touched the nine-point circle internally, their radii would be

$$r_1\frac{\cos A}{1+\cos A}, \quad r_2\frac{\cos B}{1+\cos B}, \quad r_3\frac{\cos C}{1+\cos C}.$$

7312. (By R. Tucker, M.A.)—Along the axis of a parabola ($y^2 = 9ax$) distances AK = KL = LM = a are measured from the vertex (A), and the double ordinates BKB′, CLC′, DMD′ are drawn; on DD′, lengths MR = MS = a are taken, and lines RP, SP, through any point P on CC′, meet BB′ in T, T′. Prove that perpendiculars to ST, RT′, at T, T′, meet on the parabola. [The question is susceptible of considerable generalization by assigning different lengths to AK, KL, LM, MR, MS.]

Solution by R. Knowles, B.A.; Belle Easton; *and others.*

If the coordinates of P are $(2a, y_1)$, we have the equations to RP, SP respectively $y + a = -\frac{y_1 + a}{a}(x - 3a)$, $y - a = -\frac{y_1 - a}{a}(x - 3a)$,

and, putting $x = a$, we get $Tk = 2y_1 + a$, $T'k = 2y_1 - a$;

therefore the equations to the perpendiculars to ST′, RT at the points T′, T are respectively

$$y - (2y_1 - a) = \frac{a}{y_1 - a}(x - a), \quad y - (2y_1 + a) = \frac{a}{y_1 + a}(x - a) \quad \ldots\ldots(1, 2),$$

and (1)—(2) gives for the point of intersection

$$x = \frac{y_1^2}{a}, \quad y = 3y_1, \quad \text{therefore } y^2 = 9ax,$$

and the point is on the parabola.

7302. (By Professor Townsend, F.R.S.)—If two orthogonal systems of lines in the plane of an incompressible lamina be the potential and displacement line systems of a possible irrotational strain of the lamina; show that—

(*a*) The two inverse systems of lines with respect to any arbitrary circle, real or imaginary, in the plane will be those of another possible irrotational strain of the lamina.

(*b*) At all pairs of corresponding points of the two inverse strains, the ratios of the two actual displacements and of the two principal dilatations are proportional respectively to those of the simple and those of the squared inverse distances from the centre of the circle.

Solution by the Proposer.

To prove the first part. The centre and any pair of rectangular diameters of the circle being taken as origin and axes of coordinates, it is to be shown, adopting the usual notation, that when $\phi(x, y)$ and $\psi(x, y)$ satisfy the equations $\frac{d\phi}{dx} = \frac{d\psi}{dy}$ and $\frac{d\phi}{dy} = -\frac{d\psi}{dx}$,

then $\phi\left(\frac{k^2 x'}{x'^2 + y'^2}, \frac{k^2 y'}{x'^2 + y'^2}\right)$ and $\psi\left(\frac{k^2 x'}{x'^2 + y'^2}, \frac{k^2 y'}{x'^2 + y'^2}\right)$

satisfy the equations $\frac{d\phi}{dx'} = \frac{d\psi}{dy'}$ and $\frac{d\phi}{dy'} = -\frac{d\psi}{dx'}$,

where k is the radius of the circle; which may be readily done as follows:—

Since, in all cases,

$$\frac{d\phi}{dx'} = \frac{d\phi}{dx}\cdot\frac{dx}{dx'} + \frac{d\phi}{dy}\cdot\frac{dy}{dx'}, \quad \frac{d\psi}{dx'} = \frac{d\psi}{dx}\cdot\frac{dx}{dx'} + \frac{d\psi}{dy}\cdot\frac{dy}{dx'},$$

$$\frac{d\phi}{dy'} = \frac{d\phi}{dx}\cdot\frac{dx}{dy'} + \frac{d\phi}{dy}\cdot\frac{dy}{dy'}, \quad \frac{d\psi}{dy'} = \frac{d\psi}{dx}\cdot\frac{dx}{dy'} + \frac{d\psi}{dy}\cdot\frac{dy}{dy'},$$

and since, in the present case,

$$\frac{dx}{dx'} = -k^2\frac{x'^2-y'^2}{(x'^2+y'^2)^2} = -\frac{dy}{dy'}, \quad \frac{dx}{dy'} = k^2\frac{2x'y'}{(x'^2+y'^2)^2} = \frac{dy}{dx'},$$

therefore, &c., as regards the first part.

To prove the second part. Since at once, from the above,

$$\left(\frac{d\phi}{dx'}\right)^2 + \left(\frac{d\phi}{dy'}\right)^2 = \frac{k^4}{r'^4}\left[\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right] = \frac{r^2}{r'^2}\left[\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right],$$

and since, again, from the same, after some reductions,

$$\frac{d^2\phi}{dx'^2}\cdot\frac{d^2\phi}{dy'^2} - \left(\frac{d^2\phi}{dx'\,dy'}\right)^2 = \frac{r^4}{r'^4}\left[\frac{d^2\phi}{dx^2}\cdot\frac{d^2\phi}{dy^2} - \left(\frac{d^2\phi}{dx\,dy}\right)^2\right],$$

with similar equations for ψ in each case, therefore, &c., as regards the second case.

The two orthogonal systems $r^n\cos n\theta = a^n$ and $r^n\sin n\theta = b^n$, where a and b are the parameters, in which the displacement and dilatation vary respectively as $r^{(n-1)}$ and $r^{(n-2)}$, and the two inverse systems $r_1^{-n}\cos n\theta = a_1^{-n}$ and $r_1^{-n}\sin n\theta = b_{1-n}$, where a_1 and b_1 are the parameters, in which the displacement and dilatation vary respectively as $r_1^{-(n+1)}$ and $r_1^{-(n+2)}$, supply an obvious as well as instructive example of the above general properties.

7204. (By J. Hammond, M.A.)—If

$$\phi(x) = \phi\left(\frac{1}{x\sin^2\alpha + \cos^2\alpha}\right), \quad \int_0^\infty \frac{\phi(x)\,dx}{x^2\tan^2\alpha + x - \sec^2\alpha} = 0,$$

$$\int_0^\infty \frac{\phi(x)\,dx}{(x-1)^2} = (1+\sin^2\alpha)\int_0^1 \frac{\phi(x)\,dx}{(x-1)^2}.$$

Solution by D. Edwardes; Sarah Marks; *and others.*

$$\int_0^1 \frac{\phi x}{x^2\tan^2\alpha + x - \sec^2\alpha}\,dx = -\int_1^\infty \frac{\phi y}{y^2\tan^2\alpha + y - \sec^2\alpha}\,dy$$

$$= -\int_0^\infty \frac{\phi x}{x^2\tan^2\alpha + x - \sec^2\alpha}\,dx + \int_0^1 \frac{\phi x}{x^2\tan^2\alpha + x - \sec^2\alpha}\,dx$$

if $x = (y\sin^2\alpha + \cos^2\alpha)^{-1}$,

therefore

$$\int_0^\infty \frac{\phi x}{x^2\tan^2\alpha + x - \sec^2\alpha}\,dx = 0.$$

Making the same substitution, we have

$$\int_0^1 \frac{\phi x}{(x-1)^2}\,dx = \operatorname{cosec}^2\alpha \int_1^\infty \frac{\phi y}{(y-1)^2}\,dy$$

or $$\sin^2\alpha \int_0^1 \frac{\phi x}{(x-1)^2}\,dx = \int_0^\infty \frac{\phi y}{(x-1)^2}\,dx - \int_0^1 \frac{\phi x}{(x-1)^2}\,dx,$$

therefore $$(1+\sin^2\alpha)\int_0^1 \frac{\phi x}{(x-1)^2}\,dx = \int_0^\infty \frac{\phi x}{(x-1)^2}\,dx.$$

7203. (By W. J. C. Sharp, M.A.)—In the late Professor Clifford's Quest. 6009 (*Reprint*, Vol. xxxiii., p. 59),—as to what is the form of a kite string,—assuming that the force of the wind is a function of the height, show how this function may be determined, so that the string shall assume a given form.

Solution by the Proposer.

If xy be the horizontal and vertical coordinates of any point P on the string, $\mu\,[=\phi'(y)]$ the force of the wind per vertical unit of length at that point, T the tension of the string at P, and w the weight of unit length,

$$\frac{d}{ds}\left(\mathrm{T}\frac{dx}{ds}\right) = \mu\frac{dy}{ds} = \phi'(y)\frac{dy}{ds},\quad \frac{d}{ds}\left(\mathrm{T}\frac{dy}{ds}\right) = -\mu,$$

therefore $$\mathrm{T}\frac{dx}{ds} = \phi(y) + c \quad \text{and} \quad \mathrm{T}\frac{dy}{ds} = -ws + c';$$

where c and c' are constants to be determined by the terminal conditions;

therefore $$\frac{dy}{dx}[\phi(y)+c] = c' - ws,$$

or $$[\phi(y)+c]\frac{d^2y}{dx^2} + \phi'(y)\left(\frac{dy}{dx}\right)^2 + w\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = 0,$$

which is the differential equation to the catenary. This may be identified with any given curve by substituting for $\frac{d^2y}{dx^2}, \frac{dy}{dx}$, x from the equation to that curve, and solving the resulting linear differential equation of the first order to find $\phi(y)$.

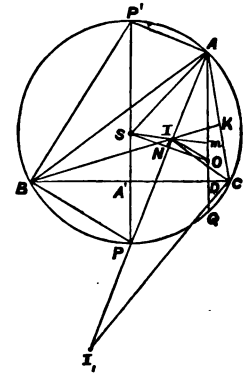
[The horizontal force acting on the element ds is the same as would act upon its vertical projection, and hence its value is μds as above.]

7179 & 7289. (7179.)—(By the Rev. G. Richardson, M.A.)—If AOD be the perpendicular from A on the opposite side of the triangle ABC, O being the orthocentre, I the centre of the inscribed circle, r its radius; prove that $IO^2 = 2r^2 - AO \cdot OD$.

(7289.)—(By R. Warburton-White, B.A.)—Show that the nine-point circle touches the inscribed and escribed circles of a triangle.

Solution by the Rev. G. Richardson, M.A.

Question 6313 is the trigonometrical equivalent of 7179, which was proposed by me in hope of obtaining a geometrical proof, and so acquiring a proof by elementary geometry of the proposition, that the nine-point circle touches the inscribed circle of a triangle. The following investigation does this, and is of a more elementary character than the proofs usually given of this theorem.



Let O be the orthocentre, S the centre of the circumscribed circle, PSP′ the diameter of the latter which bisects BC in A′. Then AP passes through I, the centre of the inscribed circle, and I_1, the centre of the escribed circle opposite to A. Draw IK (r), Im perpendicular to AC and AOD respectively. Then it is easily seen that $\angle ACI_1 = \angle AIB_1$, and consequently that the triangles AIB and ACI_1 are similar; hence we have

$$AI \cdot AI_1 = AB \cdot AC = PP' \cdot AD_1 \text{ (Euc. VI. C.)}$$

i.e., since P bisects II_1,

$$2AI \cdot AP - AI^2 = PP' \cdot AD_1$$

$$\therefore\ 2Am \cdot PP' - AI^2 = PP' \cdot AD_1 \text{ (by sim. } \Delta s.)$$

$$\therefore\ 2Am = AD + \frac{AI^2}{PP'}.$$

Now
$$IO^2 = AI^2 + AO^2 - 2AO \cdot Am \text{ (Euc. II. 13)}$$

$$= AI^2 + AO^2 - AO \cdot AD - \frac{AO \cdot AI^2}{PP'}$$

$$= AI^2 \cdot \frac{2SP - 2SA'}{PP'} - AO \cdot OD \text{ (since } AO = 2SA')$$

$$= 2AI^2 \cdot \frac{A'P}{BP} \cdot \frac{BP}{PP'} - AO \cdot OD \text{ (and by similar triangles)}$$

$$= 2AI^2 \cdot \frac{IK}{AI} \cdot \frac{IK}{AI} - AO \cdot OD = 2r^2 - AO \cdot OD.$$

Again, from the similar triangles AIK, P′PB, we have

$$PP' \cdot IK \text{ or } 2Rr = AI \cdot PB = AI \cdot IP = SA^2 - SI^2,$$

therefore
$$SI^2 = SA^2 - 2Rr = R^2 - 2Rr.$$

Also, if AD be produced to meet the circumscribed circle in Q,

$$2AO \cdot OD = AO \cdot OQ = SA^2 - SO^2,$$

therefore
$$SO^2 = R^2 - 2AO \cdot OD.$$

Finally, if N be the centre of the nine-point circle; then, since it is the middle point of SO, we have

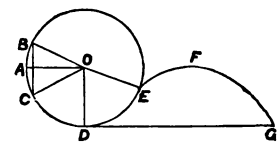
$$4NI^2 = 2SI^2 + 2IO^2 - SO^2$$

$$= 2R^2 - 4Rr + 4r^2 - 2AO \cdot OD - R^2 + 2AO \cdot OD = (R - 2r)^2,$$

hence $2NI = R - 2r$; and since R is twice the radius of the nine-point circle, this proves the theorem.

7242. (By S. Tebay, M.A.)—*Ring-leaping.* The gymnast will know the meaning of this phrase. Two hand-rings are suspended from a beam by cords. The leaper sets himself in motion by swinging to and fro, and when this has become sufficiently lively he quits the rings and alights on the ground. Find the greatest distance he can leap.

Solution by the Proposer.



Let O be the centre of the circle described by the centre of gravity of the leaper, OA a horizontal radius, OD vertical, DG a horizontal tangent. We may suppose the leaper to rise to a point B above OA; he will therefore drop vertically to C, and then descend along the arc CDE, after quitting which he will describe the parabolic arc EFG.

Let OA $= a$, $\angle \mathrm{COD} = \theta$, $\angle \mathrm{DOE} = \phi$, and v the velocity at E.

The vertical velocity at C $= \sqrt{(4ga\cos\theta)}$; the initial velocity in CDE $= \sin\theta\,(4ga\cos\theta)^{\frac{1}{2}}$, therefore

$$v^2 = 4ga\cos\theta\sin^2\theta + 2ga\,(\cos\phi - \cos\theta) = 2ga\,(t + \cos\phi),$$

putting $t = -\cos\theta\cos 2\theta$.

Now, E being origin, and y vertical, the equation to the parabolic h, after quitting the circle at E, is

$$\frac{gx^2}{2v^2\cos^2\phi} - x\tan\phi + y = 0.$$

To find the range, let $y = -a(1-\cos\phi)$, which may be taken to be approximately correct; then

$$\frac{x^2}{a\,(t+\cos\phi)\cos^2\phi} - x\tan\phi = a\,(1-\cos\phi),$$

from which we find

$$x = a\sin 2\phi\,(t+\cos\phi) + 2a\,[t\,(1+t)\cos^2\phi + (1+t)\cos^3\phi - t^2\cos^4\phi - 2t\cos^5\phi - \cos^6\phi]^{\frac{1}{2}}.$$

The range $= a\sin\phi + x$; and, θ and ϕ being independent, we must have $\delta t = 0$, or $\delta\,(-\cos\theta\cos 2\theta) = 0$, which gives

$$\cos\theta = \frac{1}{\sqrt{6}}, \quad t = \sqrt{\tfrac{2}{27}}, \text{ and } \theta = 24^\circ\,5'\,41''.$$

The range will be a max. when $\cos^3\phi + (1+2t)\cos^2\phi = 1 + t$,

that is, when $\cos^3\phi + 1{\cdot}544331\cos^2\phi = 1{\cdot}2721655$;

from which we find $\cos\phi = {\cdot}7453885$, $\phi = 41^\circ\,49'\,32''$, range $= 2{\cdot}9422a$.

Thus, if $a = 10$ feet, the range $= 9\frac{3}{4}$ yards, which is near the average.

If the leaper raise himself to A, we have $t = 0$, and $\cos^3\phi + \cos^2\phi = 1$; which gives $\cos\phi = {\cdot}75488776$. $\phi = 49^\circ$ nearly, and the range $= 2{\cdot}3935a$. Thus, if $a = 10$ feet, the range is 24 feet nearly.

[The leaper is treated as a single particle collected in the centre of gravity; otherwise the problem would present greater difficulties.]

7320. (By R. KNOWLES, B.A.) — If n_r be the coefficient of x^r in the expansion of $(1+x)^n$, prove that

$$u = n_1 - \frac{2^2-1}{2} n_2 + \frac{2^3-1}{3} n_3 - \ldots + (-1)^{n-1} \frac{2^n-1}{n} n_n$$

$$= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \ldots + (-1)^n \frac{1}{n}.$$

Solution by the Rev. D. THOMAS, M.A.; D. EASTWOOD, M.A.; *and others.*

Let $$F(n, x) = \frac{n_1 x}{1} - \frac{n_2 x^2}{2} + \ldots + (-1)^n \frac{n_n x_n}{n},$$

therefore $$F(n, x) - F(n-1, x) = \frac{1}{n} - \frac{(1-x)^n}{n},$$

$$F(n-1, x) - F(n-2, x) = \frac{1}{n-1} - \frac{(1-x)^{n-1}}{n-1},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$F(2, x) - F(1, x) = \tfrac{1}{2} - \frac{(1-x)^2}{2};$$

therefore $$F(n, x) = F(1, x) + \tfrac{1}{2} + \ldots + \frac{1}{n} - \frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} - \ldots$$

and $$u = F(n, 2) - F(n, 1)$$

$$= 2 + \tfrac{1}{2} + \ldots + \frac{1}{n} - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots - \left(1 + \tfrac{1}{2} + \ldots + \frac{1}{n}\right) = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots$$

7221. (By Professor TOWNSEND, F.R.S.)—In the irrotational strain of an incompressible substance in the space of its mass, if the equipotential surfaces of the strain be a system of confocal ellipsoids in the space, determine the form of the potential as a function of the parameter of the system.

Solution by the PROPOSER.

If ϕ be the required potential, S any surface of the confocal system, a, b, c its three semi-axes, dS its differential element at any point P, and p the perpendicular from the centre of the system on its tangent plane at P; then, by virtue of the incompressibility of the substance, from which it follows that the product of the displacement into the transverse area is constant throughout the entire extent of every displacement filament of the strain, since the product $\frac{d\phi}{dp} . d$S is constant throughout the entire extent of the displacement filament determined by dS, and since $\frac{d\phi}{dp} = \frac{d\phi}{d\lambda} \cdot \frac{d\lambda}{dp}$, where λ is the parameter of the system, therefore the product $\frac{d\phi}{\lambda d\lambda} . p d$S is constant throughout the entire extent of the filament.

But, by the familiar properties of corresponding elements in confocal ellipsoids, the product $pd\mathrm{S}$ throughout the entire extent of the filament is to the product abc in a constant ratio depending on the absolute magnitude of KS, therefore the product $\frac{d\phi}{\lambda d\lambda} \cdot abc$ is constant throughout the entire extent of the strain; hence, denoting its constant value by k, we have

$$d\phi = k\frac{\lambda d\lambda}{abc} = k\frac{\lambda d\lambda}{(a_0^2+\lambda^2)^{\frac{1}{2}}(b_0^2+\lambda^2)^{\frac{1}{2}}(c_0^2+\lambda^2)^{\frac{1}{2}}},$$

where a_0, b_0, c_0 are the semi-axes of the particular ellipsoid S_0 of the system for which $\lambda = 0$, and therefore, observing that $\phi = 0$ at infinity, we have generally

$$\phi = k\int_0^{\lambda}\frac{\lambda d\lambda}{(a_0^2+\lambda^2)^{\frac{1}{2}}(b_0^2+\lambda^2)^{\frac{1}{2}}(c_0^2+\lambda^2)^{\frac{1}{2}}};$$

a form identical, as it ought (see 6797), with that for the potential of the attraction, for the ordinary law of the inverse square of the distance, of a thin ellipsoidal shell of uniform density, bounded by coaxal and similar surfaces, throughout the entire of its containing space external to its mass.

7262. (By E. B. Elliott, M.A.)—Two bicircular curves of the fourth class have the foci of a given conic Σ for double foci, and have each a pair of tangents to Σ for double tangents. Prove that every common tangent of the two, except the tangent at the circular points, passes through one or other of two fixed points dividing harmonically the distance between the intersections of the two pairs of tangents to Σ.

Solution by Prof. Nash, M.A.; Prof. Matz, M.A.; *and others.*

Reciprocate the system with respect to S, one of the foci of Σ. Then Σ becomes a circle (C); the second focus becomes the radical axis of C and the point circle S. Each of the curves becomes a quartic having S as a double focus, the lines SI, SJ touching the curve again at points lying on the radical axis; that is, the lines SI, SJ touch the curve at the four points in which they intersect the circle C. The curve also has double points at the intersections of C with a given straight line L; hence, if $\mathrm{S} = x^2+y^2$, $\mathrm{C} = x^2+y^2-2\alpha(x-\beta)$, $\mathrm{L} = lx+my+n$, the equation of one curve is $\mathrm{C}^2 = \lambda\mathrm{SL}^2$, similarly the equation of the other curve is $\mathrm{C}^2 = \lambda'\mathrm{SL}'^2$, where $\mathrm{L}' = l'x+m'y+n'$; therefore the points of intersection of the two curves (excluding those lying on SI, SJ) lie on the straight lines $\lambda\mathrm{L}^2-\lambda'\mathrm{L}'^2=0$. These lines with L, L′ form a harmonic pencil. Therefore, &c.

6956. (By Professor Malet, M.A.)—If A, B, C, H, &c. be the minors of the discriminant of the quadric of which the quadriplanar equation is

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2+2h\alpha\beta+2g\alpha\gamma+2f\beta\gamma+2l\alpha\delta+2m\beta\delta+2n\gamma\delta = 0,$$

each taken with respect to the corresponding small letter, prove that the equations (1) of the quadric, (2) of the tangent line through $(\alpha', \beta', \gamma', \delta')$, (3) of the pair of tangent planes through the line joining the points $(\alpha', \beta', \gamma', \delta')$ and $(\alpha'', \beta'', \gamma'', \delta'')$, are given by equating to zero the respective determinants

$$\begin{vmatrix} A & H & G & L & \alpha \\ H & B & F & M & \beta \\ G & F & C & N & \gamma \\ L & M & N & D & \delta \\ \alpha & \beta & \gamma & \delta & 0 \end{vmatrix}, \qquad \begin{vmatrix} A & H & G & L & \alpha' & \alpha \\ H & B & F & M & \beta' & \beta \\ G & F & C & N & \gamma' & \gamma \\ L & M & N & D & \delta' & \delta \\ \alpha' & \beta' & \gamma' & \delta' & 0 & 0 \\ \alpha & \beta & \gamma & \delta & 0 & 0 \end{vmatrix},$$

$$\begin{vmatrix} A & H & G & L & \alpha' & \alpha'' & \alpha \\ H & B & F & M & \beta' & \beta'' & \beta \\ G & F & C & N & \gamma' & \gamma'' & \gamma \\ L & M & N & D & \delta' & \delta'' & \delta \\ \alpha' & \beta' & \gamma' & \delta' & 0 & 0 & 0 \\ \alpha'' & \beta'' & \gamma'' & \delta'' & 0 & 0 & 0 \\ \alpha & \beta & \gamma & \delta & 0 & 0 & 0 \end{vmatrix}.$$

Solution by the Proposer.

Let $\Sigma = 0$ be the condition that the plane

$$\lambda\alpha + \mu\beta + \nu\gamma + \pi\delta = 0,$$

then, if $\alpha, \beta, \gamma, \delta$ be the coordinates of the point of contact, we have

$$\frac{d\Sigma}{d\lambda} = \alpha, \quad \frac{d\Sigma}{d\mu} = \beta, \quad \frac{d\Sigma}{d\nu} = \gamma, \quad \frac{d\Sigma}{d\pi} = \delta,$$

or $\quad A\lambda + H\mu + G\nu + L\pi - \alpha = 0, \quad H\lambda + B\mu + F\nu + M\pi - \beta = 0,$

$$G\lambda + F\mu + C\nu + N\pi - \gamma = 0, \quad L\lambda + M\mu + N\nu + D\pi - \delta = 0;$$

hence, eliminating λ, μ, ν, π, we have the equation of the quadric in the required form.

Again, if the line joining $\alpha, \beta, \gamma, \delta$ to $\alpha', \beta', \gamma', \delta'$ touch the quadric, the coordinates of the point of contact of the tangent plane through it must be of the form

$$x\alpha + y\alpha', \quad x\beta + y\beta', \quad x\gamma + y\gamma', \quad x\delta + y\delta';$$

hence

$$\frac{d\Sigma}{d\lambda} - x\alpha - y\alpha' = 0, \quad \frac{d\Sigma}{d\mu} - x\beta - y\beta' = 0,$$

$$\frac{d\Sigma}{d\nu} - x\gamma - y\gamma' = 0, \quad \frac{d\Sigma}{d\pi} - x\delta - y\delta' = 0,$$

eliminating $\lambda, \mu, \nu, \pi, x, y$, we have the equation of the tangent cone in the required form.

Lastly, if the plane joining the three points

$$(\alpha, \beta, \gamma, \delta), \quad (\alpha', \beta', \gamma', \delta'), \quad (\alpha'', \beta'', \gamma'', \delta''),$$

touch the quadric, the coordinates of the point of contact must be

$$x\alpha + y\alpha' + z\alpha'', \quad x\beta + y\beta' + z\beta'', \quad x\gamma + y\gamma' + z\gamma'', \quad x\delta + y\delta' + z\delta'';$$

hence, as before, we find in the required form the equation of the pair of tangent planes through the two latter points.

7307. (By Professor WOLSTENHOLME, M.A.)—A given rigid body, of mass m, is kinetically equivalent to a system of four equal particles (each of mass $\frac{1}{4}m$) placed at the corners of any maximum tetrahedron inscribed in the ellipsoid $\dfrac{x^2}{B+C-A}+\dfrac{y^2}{C+A-B}+\dfrac{z^2}{A+B-C}=\dfrac{3}{2}$,

where MA, MB, MC are the moments of inertia about the principal axes at the centroid, which are also the axes of coordinates. If the notation, $MA = \Sigma(mx^2)$, $MB = \Sigma(my^2)$, $MC = \Sigma(mz^2)$, be used, the equation of the ellipsoid will be $\dfrac{x^2}{A}+\dfrac{y^2}{B}+\dfrac{z^2}{C}=3$.

Solution by the PROPOSER.

Let $(x_1 y_1 z_1)$, $(x_2 y_2 z_2)$, $(x_3 y_3 z_3)$, and $(x_4 y_4 z_4)$ be coordinates of four particles, each of mass $\frac{1}{4}m$; the conditions for the system, being kinetically equivalent to a given rigid body, will be

$$x_1+x_2+x_3+x_4=0=\Sigma(y)=\Sigma(z);$$

$$y_1z_1+y_2z_2+y_3z_3+y_4z_4=0=\Sigma(zx)=\Sigma(xy);$$

$$\tfrac{1}{4}m(x_1^2+x_2^2+x_3^2+x_4^2)=\Sigma(mx^2)=\tfrac{1}{4}m(B+C-A);$$

and similarly $\Sigma(y^2)=2(C+A-B)$, $\Sigma(z^2)=2(A+B-C)$.

Now, suppose we have a maximum tetrahedron inscribed in the sphere

$$x^2+y^2+z^2=1, \text{ and } (x_1y_1z_1),\ (x_2y_2z_2),\ \ldots,$$

the coordinates of its angular points, and first let $x_1=1$, $y_1=z_1=0$. Then we shall have $x_2=x_3=x_4=-\frac{1}{3}$, and therefore

$\Sigma(x)=0$, $y_2+y_3+y_4=0$, and therefore $\Sigma(y)=0$, and so $\Sigma(z)=0$.

Also $y_2z_2+y_3z_3+y_4z_4=0$, because (y_2z_2), (y_3z_3), (y_4z_4) are the coordinates of a maximum triangle inscribed in the circle $x=-\frac{1}{3}$, $y^2+z^2=\frac{8}{9}$; hence $\Sigma(yz)=0$, $\Sigma(zx)=z_2x_2+z_3x_3+z_4x_4=-\frac{1}{3}(z_2+z_3+z_4)=0$, and so $\Sigma(xy)=0$; $\Sigma(x^2)=x_1^2+x_2^2+x_3^2+x_4^2=1+\frac{1}{3}=\frac{4}{3}$; $\Sigma(y^2)-\Sigma(z^2)=0$. Again, because (y_2z_2), (y_3z_3), (y_4z_4) are corners of a maximum triangle, and $\Sigma(y^2)+\Sigma(z^2)=3\times\frac{8}{9}=\frac{8}{3}$; therefore $\Sigma(y^2)=\Sigma(z^2)=\frac{4}{3}$.

Next, refer to any other system of rectangular axes, and put

$$x'=l_1x+m_1y+n_1z,\quad y'=l_2x+m_2y+n_2z,\quad z'=l_3x+m_3y+n_3z.$$

Then $$\Sigma(x')=l_1\Sigma(x)+m_1\Sigma(y)+n_1\Sigma(z)=0,$$

and so $$\Sigma(y')=0,\quad \Sigma(z')=0;$$

$$\Sigma(y'z')=l_2l_3\Sigma(x^2)+m_2m_3\Sigma(y^2)+n_2n_3\Sigma(z^2)+(m_2n_3+m_3n_2)\Sigma(yz)$$
$$+\ldots=(l_2l_3+m_2m_3+n_2n_3)\Sigma(x^2)=0;$$

and so $$\Sigma(z'x')=\Sigma(x'y')=0;$$

and $$\Sigma(x'^2)=(l_1^2+m_1^2+n_1^2)\Sigma(x^2)=\tfrac{4}{3}=\Sigma(y'^2)=\Sigma(z'^2).$$

Thus the corners of a maximum tetrahedron inscribed in the sphere

$x^2+y^2+z^2=1$, satisfy the conditions

$$\Sigma(x)=0=\Sigma(y)=\Sigma(z),\quad \Sigma(yz)=\Sigma(zx)=\Sigma(xy)=0,$$
$$\Sigma(x^2)=\Sigma(y^2)=\Sigma(z^2)=\tfrac{1}{3};$$

and writing $\frac{x}{a}$, $\frac{y}{b}$, $\frac{z}{c}$ for (xyz), we shall have the corners of a maximum tetrahedron inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, satisfying the conditions $\Sigma(x)=\Sigma(y)=\Sigma(z)=0$, $\Sigma(yz)=\Sigma(zx)=\Sigma(xy)=0$,

$$\Sigma(x^2)=\tfrac{1}{3}a^2,\quad \Sigma(y^2)=\tfrac{1}{3}b^2,\quad \Sigma(z^2)=\tfrac{1}{3}c^2,$$

and making $a^2=\frac{3}{2}(B+C-A)$, $b^2=\frac{3}{2}(C+A-B)$, $c^2=\frac{3}{2}(A+B-C)$, we get the conditions required for kinetic equivalence to be as stated.

7218. (By Prof. Genese, M.A.)—Prove that

$$\begin{vmatrix} x & 0 & 0 & x_1^2 & x_2^2 & l \\ 0 & y & 0 & y_1^2 & y_2^2 & m \\ 0 & 0 & z & z_1^2 & z_2^2 & n \\ 0 & z & y & 2y_1z_1 & 2y_2z_2 & 2p \\ z & 0 & x & 2z_1x_1 & 2z_2x_2 & 2q \\ y & x & 0 & 2x_1y_1 & 2x_2y_2 & 2r \end{vmatrix} = -2\Delta\{lX_1X_2+\ldots+p(Y_1Z_2+Z_1Y_2)+\ldots\}$$

where $\Delta=\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$, X_1 is the minor of x_1 in Δ, &c.

7217. (By T. Muir, M.A., F.R.S.E.)—Prove the identity

$$\begin{vmatrix} x & 0 & 0 & x_1^2 & x_1x_2 & x_2^2 \\ 0 & y & 0 & y_1^2 & y_1y_2 & y_2^2 \\ 0 & 0 & z & z_1^2 & z_1z_2 & z_2^2 \\ 0 & z & y & 2y_1z_1 & y_1z_2+y_2z_1 & 2y_2z_2 \\ z & 0 & x & 2z_1x_1 & z_1x_2+x_1z_2 & 2z_2x_2 \\ y & x & 0 & 2x_1y_1 & x_1y_2+x_2y_1 & 2x_2y_2 \end{vmatrix} = |xy_1z_2|^3.$$

Solution by G. Heppel, M.A.; Professor Roy, M.A.; *and others.*

(7218). By suitably interchanging rows and columns, it may be shown that l, m, n, and p, q, r, are symmetrically involved with x, y, z.

$$\text{Coeff. of } l = z\begin{vmatrix} 0 & x & 2x_1y_1 & 2x_2y_2 \\ 0 & y & y_1^2 & y_2^2 \\ z & 0 & z_1^2 & z_2^2 \\ y & z & 2y_1z_1 & 2y_2z_2 \end{vmatrix} - y\begin{vmatrix} 0 & y & 2x_1z_1 & 2x_2z_2 \\ 0 & z & z_1^2 & z_2^2 \\ y & 0 & y_1^2 & y_2^2 \\ z & y & 2y_1z_1 & 2y_2z_2 \end{vmatrix}$$

$$\text{First term} = z^2\begin{vmatrix} x & 2x_1y_1 & 2x_2y_2 \\ y & y_1^2 & y_2^2 \\ z & 2y_1z_1 & 2y_2z_2 \end{vmatrix} - yz\begin{vmatrix} x & 2x_1y_1 & 2x_2y_2 \\ y & y_1^2 & y_2^2 \\ 0 & z_1^2 & z_2^2 \end{vmatrix}$$

$$= 2y_1y_2z^2\Delta - xyz\begin{vmatrix} y_1^2 & y_2^2 \\ z_1^2 & z_2^2 \end{vmatrix} + 2z^2y\begin{vmatrix} y_1z_1 & y_2z_2 \\ x_1y_1 & x_2y_2 \end{vmatrix} - 2y^2z\begin{vmatrix} z_1^2 & z_2^2 \\ x_1y_1 & x_2y_2 \end{vmatrix}$$

$$= 2y_1y_2z^2\Delta - yz\left\{(y\ z_2+y_2z_1)\,x\mathrm{X}+2\begin{vmatrix} z_1\mathrm{X}_2 & -z_2\mathrm{X}_1 \\ x_1y_1 & x_2y_2\end{vmatrix}\right\}$$

$$= 2y_1y_2z^2\Delta - yz\,[(y_1z_2+y_2z_1)\,x\mathrm{X}+2y_2z_1x_2\mathrm{X}_2+2y_1z_2x_1\mathrm{X}_1].$$

Hence, introducing the second term, and remembering that interchanging y and z alters the signs of Δ, X, X_1, X_2, we have

Coefficient of $l = 2\,(y_1y_2z^2+z_1z_2y^2)\,\Delta - 2\,(y_1z_2+y_2z_1)\,yz\,(x\mathrm{X}+x_1\mathrm{X}_1+x_2\mathrm{X}_2)$

$$= -2\mathrm{X}_1\mathrm{X}_2\Delta.$$

The coefficient of $2p$ is $-\begin{vmatrix} x & 0 & z & 2z_1x_1 & 2z_2x_2 \\ 0 & x & y & 2x_1y_1 & 2x_2y_2 \\ 0 & 0 & x & x_1^2 & x_2^2 \\ 0 & y & 0 & y_1^2 & y_2^2 \\ z & 0 & 0 & z_1^2 & z_2^2 \end{vmatrix}$.

$$= -x^3\begin{vmatrix} y_1^2 & y_2^2 \\ z_1^2 & z_2^2\end{vmatrix} + xy^2\begin{vmatrix} z_1^2 & z_2^2 \\ x_1^2 & x_2^2\end{vmatrix} + xz^2\begin{vmatrix} x_1^2 & x_2^2 \\ y_1^2 & y_2^2\end{vmatrix}$$

$$-2x^2y\begin{vmatrix} z_1^2 & z_2^2 \\ x_1y_1 & x_2y_2\end{vmatrix} + 2x^2z\begin{vmatrix} y_1^2 & y_2^2 \\ x_1z_1 & x_2z_2\end{vmatrix} + yzx_1x_2\begin{vmatrix} z & 2z_1 & 2z_2 \\ y & 2y_1 & 2y_2 \\ x & x_1 & x_2\end{vmatrix}.$$

Let $\quad y_1z_2+y_2z_1 = \mathrm{K},\quad z_1x_2+z_2x_1 = \mathrm{L},\quad x_1y_2+x_2y_1 = \mathrm{M},$

then the following relations may be shown

$$\begin{vmatrix} x_1^2 & x_2^2 \\ y_1^2 & y_2^2\end{vmatrix} = \mathrm{MZ},\quad 2\begin{vmatrix} y_1^2 & y_2^2 \\ x_1z_1 & x_2z_2\end{vmatrix} = \mathrm{MX}-\mathrm{KZ},$$

$$-2\begin{vmatrix} z_1^2 & z_2^2 \\ x_1y_1 & x_2y_2\end{vmatrix} = \mathrm{LX}-\mathrm{KY},\quad \mathrm{LZ}+\mathrm{MY} = -2x_1x_2\mathrm{X},$$

and lastly $\quad \mathrm{Y}_1\mathrm{Z}_2+\mathrm{Y}_2\mathrm{Z}_1 = x^2\mathrm{K}-xy\,\mathrm{L}-xz\,\mathrm{M}+2yzx_1x_2.$

Hence coeff. of $2p = -x^3\mathrm{KX}+xy^2\mathrm{LY}+xz^2\mathrm{MZ}+x^2y\,(\mathrm{LX}-\mathrm{KY})$

$$+x^2z\,(\mathrm{MX}-\mathrm{KZ})-2yzx_1x_2\Delta+xyz\,(\mathrm{LZ}+\mathrm{MY})$$

$$= -(x\mathrm{X}+y\mathrm{Y}+z\mathrm{Z})(x^2\mathrm{K}-xy\,\mathrm{L}-xz\,\mathrm{M})-2yzx_1x_2\Delta$$

$$= -(\mathrm{Y}_1\mathrm{Z}_2-\mathrm{Y}_2\mathrm{Z}_1)\,\Delta.$$

[The determinant equated to zero is the equation to the locus of pairs of points conjugate with respect to the conics which pass through two points (x_1, y_1, z_1), (x_2, y_2, z_2), and have their coefficients connected by a linear relation; of which locus $\Delta = 0$, the equation to the line joining the points, is clearly a factor.]

(7217). This is derived at once from 7218, by interchanging the last two columns of the determinant in 7217, which will thus appear to be equal to the sum of three symmetrical expressions, of which one is

$$[2x_1x_2\mathrm{X}_1\mathrm{X}_2+(y_1z_2+y_2z_1)\,(\mathrm{Y}_1\mathrm{Z}_2+\mathrm{Y}_2\mathrm{Z}_1)]\,\Delta,$$

and this sum resolves into the two products

$$(x_1\mathrm{X}_1+y_1\mathrm{Y}_1+z_1\mathrm{Z}_1)(x_2\mathrm{X}_2+y_2\mathrm{Y}_2+z_2\mathrm{Z}_2)\,\Delta$$

$$+(x_1\mathrm{X}_2+y_1\mathrm{Y}_2+z_1\mathrm{Z}_2)(x_2\mathrm{X}_1+y_2\mathrm{Y}_1+z_2\mathrm{Z}_1)\,\Delta,$$

and each factor of the first $= \Delta$; two factors of the second $= 0$; therefore determinant $= \Delta^3$.

7303. (By Professor MOREL.)—Si A, B, C, a, b, c sont les éléments d'un triangle, et ϕ un angle satisfaisant à la relation

$$\frac{b^2+c^2-2bc\cos(A-2\phi)}{b^2+c^2-2bc\cos A} = \frac{c^2+a^2-2ca\cos(B-2\phi)}{c^2+a^2-2ca\cos C};$$

(1) l'un quelconque de ces rapports est aussi égal à

$$\frac{a^2+b^2-2ab\cos(C-2\phi)}{a^2+b^2-2ab\cos C};$$

et en outre (2) on a, en supprimant la solution $\phi = 0$, la relation

$$\cot\phi = \cot A + \cot B + \cot C.$$

Solution by W. H. BLYTHE, M.A.; Rev. D. THOMAS, M.A.; *and others.*

The given equations reduce to $b^3\sin\phi\sin(A-\phi) = a^3\sin\phi\sin(B-\phi)$,

hence $\sin\phi = 0$, or else $\cot\phi = \dfrac{b^3\cos A - a^3\cos B}{b^3\sin A - a^3\sin B}$;

and, putting $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, and $\sin A = \dfrac{a}{2R}$, &c.,

we get $\cot\phi = \dfrac{2R}{abc}(b^2+a^2+c^2) = \cot A + \cot B + \cot C.$

By symmetry, this value must also satisfy the equation formed by putting either of the first-given expressions equal to the third.

7091. (By W. J. C. SHARP, M.A.)—Show that (1) if the decimal equivalent to a fraction whose denominator is n have the greatest possible number of figures $(n-1)$ in the period, this is the same for all such fractions; (2) when this is the case, n is a prime number; and (3) no fraction with a denominator n can have $n-2$ figures in the period of the corresponding decimal.

Solution by the PROPOSER.

If the decimal equivalent to $\frac{1}{n}$ has $n-1$ figures in the period, the whole $n-1$ possible remainders in the division of 1 by n have been brought down, and consequently $\frac{2}{n}$, $\frac{3}{n}$, &c. give the same strings of figures as occur after the remainders 2, 3, &c. have been brought down in $\frac{1}{n}$. $\frac{1}{7}$, $\frac{2}{7}$, &c., $\frac{1}{17}$, $\frac{2}{17}$, &c., are examples.

Again, if $n = pqr\ldots$, where p, q, &c. are odd primes (even numbers evidently need not be considered), $\frac{a}{p} + \frac{b}{q} + \ldots = \frac{l}{n}$, and the number of figures in the period cannot exceed the L. C. M. of $p-1$, $q-1$, &c., which is less than $n-2$.

In (3), the denominator 3 should have been excepted. From what is said above, n must be prime if there are $n-2$ figures in the period, and

therefore $10^{n-2}-1$ must be divisible by n; but $10^{n-1}-1$ is so in this case, and therefore $9 \times 10^{n-2}$ must be so, which can only be the case when $n=3$ and $n=2$ and $n=5$ give terminating decimals, and are not prime to 10.

5820. (By Professor SYLVESTER, F.R.S.)—If $x^3+y^3+z^3 = 3\mu xyz$, and $x^6+y^6+z^6-x^3z^3-z^3y^3-y^3x^3 = 0$; show that (1) the variables will bear 18 distinct ratios to each other, if their order be taken into account; and (2) if their order be disregarded, these are reducible to the three following ratios:—

$$(1+q)^{\frac{1}{3}} : (1+q\rho)^{\frac{1}{3}} : (1+q\rho^2)^{\frac{1}{3}}, \quad (1+q)^{\frac{1}{3}} : (1+q\rho)^{\frac{1}{3}}\,\rho : (1+q\rho^2)^{\frac{1}{3}}\,\rho^2,$$

$$(1+q)^{\frac{1}{3}} : (1+q\rho)^{\frac{1}{3}}\,\rho^2 : (1+q\rho^2)^{\frac{1}{3}}\,\rho,$$

where $$q = \left(\frac{27}{\mu^3}-1\right)^{\frac{1}{3}} \text{ and } \rho^3 = 1.$$

[Professor SYLVESTER remarks that this Question contains virtually the complete analytical solution of the problem of finding the points in which a given cubic is osculable by other cubics in 9 consecutive points.]

Solution by W. J. C. SHARP, M.A.

If x^3, y^3, z^3 be the roots of $t^3-pt^2+q't-r = 0$, we have, from the given relations, $$p^3 = 27\mu^3 r \text{ and } p^2 = 3q',$$

and the equation becomes $$t^3-pt^2+\frac{p^2}{3}\,t-\frac{p^3}{27\mu^3} = 0,$$

or, if $$u = \frac{3t}{p}, \quad u^3-3u^2+3u-\frac{1}{\mu^3} = 0,$$

$$\therefore (u-1)^3 = \mu^{-3}-1 = q^3 \text{ suppose, } \therefore u = 1+q,\ 1+q\rho, \text{ or } 1+q\rho^2,$$

where ρ is an imaginary cube-root of unity, and x^3, y^3, z^3 are proportional to these quantities in some order, and, since the product of the three values of $u = q^3+1\ (=\mu^{-3})$, and only those values are to be chosen which make the cube root of this product μ^{-1}; the relations are as given in the Question.

If regard be paid to the order, each of these ratios will represent six different solutions, and therefore (1) is true.

7327. (By Professor WOLSTENHOLME, M.A.) — Prove that (1) the maximum value of $f(x) \equiv (1+x)^{\frac{1}{2}(1+x)}\,(1-x)^{\frac{1}{2}(1-x)}+(1-x^2)^{\frac{1}{2}}-2$ lies between ·093511 and ·093512; and (2) the corresponding value of x is ·9653863 very nearly.

Solution by G. HEPPEL, M.A.; Prof. NASH, M.A.; *and others.*

Let $f(x) = z$, $(1+x)^{\frac{1}{2}(1+x)} = s$, $(1-x)^{\frac{1}{2}(1-x)} = t$;

then $z = st + (1-x^2)^{\frac{1}{2}} - 2$, $\frac{ds}{dx} = \frac{s}{2}[1 + \log(1+x)]$,

$\frac{dt}{dx} = -\frac{t}{2}[1 + \log(1-x)]$; therefore $\frac{d}{dx}.(st) = \frac{st}{2}\log\frac{1+x}{1-x}$;

hence $\frac{dz}{dx} = (1-x^2)^{\frac{1}{2}}.\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}x}.\frac{1}{2}\log\frac{1+x}{1-x} - \frac{x}{(1-x^2)^{\frac{1}{2}}}$,

and, if z be a maximum, we have

$$(1-x^2)\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}x}.\frac{1}{2}\log\frac{1+x}{1-x} - x = 0.$$

Put $\frac{1+x}{1-x} = u$, then $x = \frac{u-1}{u+1}$, $1-x^2 = \frac{4u}{(u+1)^2}$,

and the condition becomes

$$v = \frac{2u}{(u+1)^2}.u^{\frac{u-1}{2(u+1)}}.\log u - \frac{u-1}{u+1} = 0.$$

By rough approximations, it is easy to see that u must be between 56 and 57. Calculating more accurately, we find that $u = 56$ makes $v = \cdot0027886$, and $u = 57$ makes $v = -\cdot0007518$. This leads to the closer approximation $u = 56\cdot79$, which gives $v = -\cdot0000320$. From these results we get the corrected value of u as $56\cdot780664$, and that makes x between $\cdot9653862$ and $\cdot9653863$. The value of z may then be at once calculated, being between $\cdot093511$ and $\cdot093512$.

6971 & 7087. (By G. S. CARR, M.A.)—(6971.) If a number of voltaic cells differing both in electro-motive force and conductivity be joined "in multiple arc," show that the difference of potentials of the electrodes of the battery, when connected by a wire, is equal to the sum of the current strengths of the separate cells divided by the sum of the conductivities of all parts of the circuit.

(7087.)—If a number of voltaic elements, all differing in electro-motive force and resistance, be arranged in unequal groups, the cells of each group in multiple arc and the groups in a series; show that the current which passes through a wire connecting the poles of the battery will be given by the following rule. Multiply the sum of the separate current strengths of the cells in each group by the effective resistance of the group, and divide the sum of all the products so formed by the total effective resistance of the circuit.

Solution by the PROPOSER.

Let there be a series of three groups (AB, BC, CD) of cells in multiple arc, the poles AD being connected by a wire of resistance R, and the total current being C. Let the electro-motive forces of the cells of AB be

e_1, e_1', e_1'', &c., and the conductivities k_1, k_1', k_1'', &c. Let the sum of the currents which could be obtained from these cells if detached be c_1; then

$$c_1 = k_1e_1 + k_1'e_1' + \&c. = \Sigma k_1e_1.$$

Similarly for BC and CD, $c_2 = \Sigma k_2e_2$, $c_3 = \Sigma k_3e_3$. Also let the whole resistance of the multiple arc AB be ρ_1, so that

$$\rho_1^{-1} = k_1 + k_1' + \&c. = \Sigma k_1. \quad \text{Similarly } \rho_2^{-1} = \Sigma k_2,\ \rho_3^{-1} = \Sigma k_3.$$

Let the potentials at A, B, C, D, the points of connection, be 0, x, y, z. Now at each of these points the current will be C; therefore, by adding the actual currents of the divided arc AB, we shall have

$$C = \Sigma k_1(e_1 - x) = \Sigma k_1e_1 - x\Sigma k_1 = c_1 - x\rho_1^{-1}.$$

Similarly, from BC and CD, $C = c_2 + (x-y)\rho_2^{-1}$, $C = c_3 + (y-z)\rho_3^{-1}$.

By eliminating x and y from these equations, we obtain

$$z + C(\rho_1 + \rho_2 + \rho_3) = c_1\rho_1 + c_2\rho_2 + c_3\rho_3.$$

But $z = RC$; therefore $C = \dfrac{c_1\rho_1 + c_2\rho_2 + c_3\rho_3}{R + \rho_1 + \rho_2 + \rho_3}$ or $\dfrac{\Sigma(c\rho)}{R + \Sigma(\rho)}$,

which is equivalent to the rule enunciated in (7087), and the proof is obviously general. Next, let there be but one multiple arc AB; then, by the formula above, $C = \dfrac{c_1\rho_1}{R + \rho_1}$, therefore $E = CR = \dfrac{c_1}{\dfrac{1}{\rho_1} + \dfrac{1}{R}}$, which is equivalent to the rule in (6971).

[Another solution of Quest. 6971 is given on p. 104 of Vol. 38 of *Reprints*.]

7330. (By the Editor.) — Prove that the area of a triangle whose sides are roots of the equation $f(x) \equiv x^3 + px^2 + qx + r = 0$, is $\Delta = \frac{1}{4}p^{\frac{1}{2}}(4pq - p^3 - 8r)^{\frac{1}{2}}$.

Solution by Harold Harley; W. W. Taylor, M.A.; *and others.*

If x_1, x_2, x_3 are the roots of $f(x)$, we have

$$\Delta^2 = s(s-x_1)(s-x_2)(s-x_3).$$

Now $(s-x_1)(s-x_2)(s-x_3) = f(s) = s^3 + ps^2 + qs + r$, and $s = -\frac{1}{2}p$.

Therefore $\Delta = [-\frac{1}{2}p(-\frac{1}{8}p^3 + \frac{1}{4}p^3 - \frac{1}{2}pq + r)]^{\frac{1}{2}} = \frac{1}{4}p^{\frac{1}{2}}(4pq - p^3 - 8r)^{\frac{1}{2}}$.

[If Δ be the area, and a, b, c the sides, the result comes from $16\Delta^2 = (a^2 + b^2 + c^2)^2 - 2(a^4 + b^4 + c^4) = (p^2 - 2q)^2 - 2(p^4 - 4p^2q + 4pr + 2q^2)$.]

6343. (By Professor Tanner, M.A.)—Solve the equations

$$2u_nv_n = 2u_{n+1}^2 = (u_n + u_{n+1})v_{n+1}.$$

Solution by J. HAMMOND, M.A.; BELLE EASTON; *and others.*

Since $v_n = \frac{u_{n+1}^2}{u_n}$, we have $2u_{n+1}^2 = (u_n + u_{n+1})\frac{u_{n+2}^2}{u_{n+1}}$ ..... (1).

In (1) put $u_n = u_{n+1}w_n$, then we have

$$2u_{n+2}^2 w_{n+1}^2 = (u_{n+1}w_n + u_{n+1})\frac{u_{n+2}^2}{u_{n+1}}, \text{ or } 2w_{n+1}^2 = w_n + 1 \text{ .....(2).}$$

The solution of (2) is $w_n = \cos\left(\frac{\alpha}{2^n}\right)$, where α is an arbitrary periodic constant; hence

$$u_{n+1} = u_n \sec\left(\frac{\alpha}{2^n}\right), \text{ or } u_n = C\sec\left(\frac{\alpha}{2^{n-1}}\right)\sec\left(\frac{\alpha}{2^{n-2}}\right) \text{ .....;}$$

and $v_n = \sec\left(\frac{\alpha}{2^n}\right)u_{n+1}$, or $v_n = C\sec^2\left(\frac{\alpha}{2^n}\right)\sec\left(\frac{\alpha}{2^{n-1}}\right)\sec\left(\frac{\alpha}{2^{n-2}}\right)\cdots$

From those given may be deduced the somewhat simpler solutions

$$u_n = 2^n \sin\left(\frac{\beta}{2^n}\right)A, \quad v_n = 2^{n+1}\tan\left(\frac{\beta}{2^{n+1}}\right)A.$$

6197. (By Professor MOREL.)—Résoudre $x^{\frac{1}{2}} - x^{\frac{1}{3}} - 7x^{\frac{1}{6}} + x^{\frac{1}{2}} + 6 = 0$.

Solution by J. HAMMOND, M.A.; J. O'REGAN; *and others.*

Putting $x = y^6$, we have

$$y^4 - y^3 - 7y^2 + y + 6 = (y^2 - 1)(y^2 - y - 6) = 0;$$

whence $y = -1, 1, -2, 3$, and $x = 1, 1, 2^6, 3^6$.

5618. (By Professor SEITZ, M.A.)—The perimeter of an inscriptible quadrilateral is $2a$; show that (1) its average area is $\frac{1}{8}\pi a^2\left(\frac{6\,8}{10\,5} - \frac{1}{2}\pi\right)$; (2) if the quadrilateral is also circumscriptible, its average area is $\frac{1}{8\,4}\pi^2 a^2$; and (3) the average area of the inscribed circle is $\frac{1}{16}\pi a^2$.

Solution by the PROPOSER.

1. Let ABCD be the quadrilateral, and put AB = x, BC = y, CD = z, DA = $2a - x - y - z$, area ABCD = u; then

$$u = [(a-x)(a-y)(a-z)(x+y+z-a)]^{\frac{1}{2}};$$

the limits of x are 0 and a; from $y = 0$ to $y = a - x = s$ the limits of z are $a - x - y = t$ and a; and from $y = s$ to $y = a$ the limits of z are 0 and

$2a-x-y=v$; hence the required average is

$$\Delta_1=\int_0^a\left\{\int_0^s\int_t^a u\,dy\,dz+\int_s^a\int_0^v u\,dy\,dz\right\}dx+\int_0^a\left\{\int_0^s\int_t^a dy\,dz+\int_s^a\int_0^v dy\,dz\right\}dx$$

$$=\frac{3}{2a^3}\int_0^a\left\{\int_0^s\int_t^a u\,dy\,dz+\int_s^a\int_0^v u\,dy\,dz\right\}dx$$

$$=\frac{3\pi}{16a^3}\int_0^a\int_0^{a-x}(x+y)^2[(a-x)(a-y)]^{\frac{1}{2}}\,dx\,dy$$

$$+\frac{3}{8a^3}\int_0^a\int_{a-x}^a\left\{(x+y)^2[(a-x)(a-y)]^{\frac{1}{2}}\sin^{-1}\left(\frac{2a}{x+y}-1\right)\right.$$
$$\left.+2\,(2a-x-y)\,[a\,(a-x)(a-y)(x+y-a)]^{\frac{1}{2}}\right\}dx\,dy$$

$$=\frac{\pi}{280a^3}\int_0^a\left\{(8a^2+28ax+35x^2)[a\,(a^2-ax)^{\frac{1}{2}}-x\,(ax-x^2)^{\frac{1}{2}}]\right\}dx$$

$$+\frac{3}{8a^3}\int_0^a\int_{a-x}^a\left\{(x+y)^2[(a-x)(a-y)]^{\frac{1}{2}}\sin^{-1}\left(\frac{2a}{x+y}-1\right)\right.$$
$$\left.+2\,(2a-x-y)\,[a\,(a-x)(a-y)(x+y-a)]^{\frac{1}{2}}\right\}dx\,dy.$$

Put $x+y=a+x\sin^2\phi$; then

$$a-y=x\cos^2\phi,\quad x+y-a=x\sin^2\phi,$$

$$\sin^{-1}\left(\frac{2a}{x+y}-1\right)=\sin^{-1}\left(\frac{a-x\sin^2\phi}{a+x\sin^2\phi}\right),\quad dy=2x\sin\phi\cos\phi\,d\phi,$$

when $y=a-x$, $\phi=0$, and when $y=a$, $\phi=\frac{1}{2}\pi$. Therefore

$$\Delta_1=\frac{\pi}{280a^3}\int_0^a\left\{(8a^2+28ax+35x^2)\,[a\,(a^2-ax)^{\frac{1}{2}}-x\,(ax-x^2)^{\frac{1}{2}}]\right\}dx$$

$$+\frac{3}{4a^3}\int_0^a\int_0^{\frac{1}{2}\pi}\left\{x\,(a+x\sin^2\phi)^2\,(ax-x^2)^{\frac{1}{2}}\sin^{-1}\left(\frac{a-x\sin^2\phi}{a+x\sin^2\phi}\right)\right.$$
$$\left.+2x^2\,(a-x\sin^2\phi)(a^2-ax)^{\frac{1}{2}}\sin\phi\right\}\sin\phi\cos^2\phi\,dx\,d\phi$$

$$=\frac{\pi}{280a^3}\int_0^a\left\{(24a^3+84a^2x+105ax^2)(a^2-ax)^{\frac{1}{2}}-16\,(a+x)^3\,(a^2-x^2)^{\frac{1}{2}}\right\}dx$$
$$=\tfrac{1}{8}\pi a^2\left(\tfrac{6}{10}\tfrac{5}{5}-\tfrac{1}{8}\pi\right).$$

2. Put $\mathrm{AB}=x$, $\mathrm{BC}=y$, $\mathrm{CD}=a-x$, $\mathrm{DA}=a-y$.
Then $u=[(ax-x^2)(ay-y^2)]^{\frac{1}{2}}$, and the required average is

$$\Delta_2=\int_0^a\int_0^a u\,dx\,dy+\int_0^a\int_0^a dy\,dz=\frac{1}{a^2}\int_0^a\int_0^a[(ax-x^2)(ay-y^2)]^{\frac{1}{2}}dx\,dy=\tfrac{1}{64}\pi^2a^2.$$

3. Let $r=$ the radius of the inscribed circle. Then $r=\frac{u}{a}$, and the required average is

$$\Delta_3=\int_0^a\int_0^a\pi r^2\,dx\,dy+\int_0^a\int_0^a dx\,dy=\frac{\pi}{a^4}\int_0^a\int_0^a(ax-x^2)(ay-y^2)\,dx\,dy=\tfrac{1}{36}\pi a^2.$$

7283. (By Professor MARTIN, M.A., Ph.D.)—From a full cask of wine a quantity is drawn off at random, and the cask filled up with water. Find the chance that, after n such operations, the cask contains more wine than water.

Solution by the PROPOSER.

Let $x_1, x_2, x_3, \ldots, x_n$ denote the quantities of the liquid taken out at the 1st, 2nd, 3rd, ..., nth drawings, a = the capacity of the cask, and let one-m^{th} be the *part* of the contents the wine must exceed.

Then $a\left(1-\frac{x_1}{a}\right)\left(1-\frac{x_2}{a}\right)\left(1-\frac{x_3}{a}\right)\ldots\left(1-\frac{x_n}{a}\right)$ = quantity of wine remaining after the nth drawing, which must exceed $\frac{a}{m}$. Putting

$$a\left(1-\frac{x_1}{a}\right)\left(1-\frac{x_2}{a}\right)\left(1-\frac{x_3}{a}\right)\ldots\left(1-\frac{x_n}{a}\right)=\frac{a}{m},$$ we have

$$x_1 = a\left\{1-\frac{1}{m\left(1-\frac{x_2}{a}\right)\left(1-\frac{x_3}{a}\right)\left(1-\frac{x_4}{a}\right)\ldots\left(1-\frac{x_n}{a}\right)}\right\}=a_1.$$

The greater x_1 is, the less wine will be left; therefore x_1 may have any value from 0 to a_1. Therefore

$$p_1=\frac{a_1}{a}=1-\frac{1}{m\left(1-\frac{x_2}{a}\right)\left(1-\frac{x_3}{a}\right)\left(1-\frac{x_4}{a}\right)\ldots\left(1-\frac{x_n}{a}\right)}$$

is the probability on the supposition that $x_2, x_3, x_4, \ldots x_n$ are known.

The greatest value x_r can have is when $x_1, x_2, x_3, x_4, \ldots x_{r-1}$ each = 0, and the limits of $x_r, x_2, x_3, \ldots, x_n$, are 0 and (respectively)

$$a\left\{1-\frac{1}{m\left(1-\frac{x_{r+1}}{a}\right)\left(1-\frac{x_{r+2}}{a}\right)\left(1-\frac{x_{r+3}}{a}\right)\ldots\left(1-\frac{x_n}{a}\right)}\right\}=a_r,$$

$$a\left\{1-\frac{1}{m\left(1-\frac{x_2}{a}\right)\left(1-\frac{x_4}{a}\right)\left(1-\frac{x_5}{a}\right)\ldots\left(1-\frac{x_n}{a}\right)}\right\}=a_2,$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$a\left(1-\frac{1}{m}\right),=a_n.$$

Hence probability is $p_n=\dfrac{\int_0^{a_n}\ldots\int_0^{a_4}\int_0^{a_3}\int_0^{a_2}p_1\,dx_n\ldots dx_4\,dx_3\,dx_2}{\int_0^{a}\ldots\int_0^{a}\int_0^{a}\int_0^{a}dx_n\ldots dx_4\,dx_3\,dx_2}$

$$=\frac{1}{a^{n-1}}\int_0^{a_n}\ldots\int_0^{a_4}\int_0^{a_3}\int_0^{a_2}\left\{1-\frac{1}{m\left(1-\frac{x_2}{a}\right)\left(1-\frac{x_3}{a}\right)\left(1-\frac{x_4}{a}\right)\ldots\left(1-\frac{x_n}{a}\right)}\right\}\times dx_n\ldots dx_4\,dx_3\,dx_2.$$

1. Let $n = 2$, then we have

$$p_2 = \frac{1}{a}\int_0^{a_2}\left\{1-\frac{1}{m\left(1-\frac{x_2}{a}\right)}\right\}dx_2, = \frac{1}{a}\left[x_2+\frac{a}{m}\log\left(1+\frac{x_2}{a}\right)\right]_0^{a_2},$$

$$= 1-\frac{1}{m}(1+\log m).$$

2. Let $n = 3$, then we have

$$p_3 = \frac{1}{a^2}\int_0^{a_3}\int_0^{a_2}\left\{1-\frac{1}{m\left(1-\frac{x_2}{a}\right)\left(1-\frac{x_3}{a}\right)}\right\}dx_3\,dx_2,$$

$$= \frac{1}{a}\int_0^{a_3}\left[\left\{1-\frac{1}{m\left(1-\frac{x_3}{a}\right)}\right\}+\frac{1}{m\left(1-\frac{x_3}{a}\right)}\log\left\{\frac{1}{m\left(1-\frac{x_3}{a}\right)}\right\}\right]dx_3,$$

$$= \frac{1}{a}\left[x_3+\frac{a}{m}\log\left(1-\frac{x_3}{a}\right)+\frac{a}{2m}\left\{\log\left[m\left(1-\frac{x_3}{a}\right)\right]\right\}^2\right]_0^{a_3},$$

$$= 1-\frac{1}{m}\left[1+\log m+\frac{1}{1\,.\,2}(\log m)^2\right].$$

3. Let $n = 4$, then we have

$$p_4 = \frac{1}{a^3}\int_0^{a_4}\int_0^{a_3}\int_0^{a_2}\left\{1-\frac{1}{m\left(1-\frac{x_2}{a}\right)\left(1-\frac{x_3}{a}\right)\left(1-\frac{x_4}{a}\right)}\right\}dx_4\,dx_3\,dx_2,$$

$$= 1-\frac{1}{m}\left[1+\log m+\frac{1}{1\,.\,2}(\log m)^2+\frac{1}{1\,.\,2\,.\,3}(\log m)^3\right].$$

Generally, $p_n = 1-\frac{1}{m}\left[1+\log m+\frac{1}{1\,.\,2}(\log m)^2+\frac{1}{1\,.\,2\,.\,3}(\log m^3)+\ldots\right.$

$$\left.+\ldots\frac{1}{1\,.\,2\,.\,3\ldots n-1}(\log m)^{n-1}\right]$$

$$= 1-\frac{1}{m}(e_n^{\log m}), \quad = 1-\tfrac{1}{2}(e_n^{\log 2}), \text{ when } m = 2.$$

When n is infinite, $p_n = 1-\frac{e^{\log m}}{m} = 1-\frac{m}{m} = 0$, as it should be.

7258. (By the Editor.)—If a square board, of side a, be supported in a horizontal position by two equal loops of string, of length λ, that pass under opposite corners, and are suspended from a fixed hook; prove that the tensions of the strings are

$$\frac{W}{4}\cdot\frac{\lambda-a\sqrt{2}}{(\lambda^2-2\sqrt{2}a\lambda)^{\frac{1}{2}}}.$$

Solution by W. M. Coates, M.A.; R. W. White, M.A.; *and others.*

Let ABCD be the board [the figure can be easily imagined], G its centre of gravity, O the point of suspension, and T the tension of the string; then we have

$$OA = \frac{\lambda - a\sqrt{2}}{2}, \quad AG = \frac{a}{\sqrt{2}}, \quad OG = \frac{(\lambda^2 - 2\sqrt{2}a\lambda)^{\frac{1}{2}}}{2}, \quad \text{and } 4T\frac{OG}{OA} = W;$$

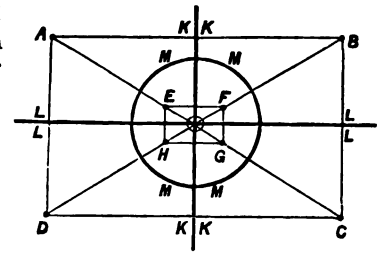
hence the stated result follows.

7122. (By Professor Townsend, F.R.S.)—A point, taken arbitrarily in the plane of a tetrad of circles, every two of which intersect at right angles, and three of which are consequently real while the fourth is imaginary, being supposed to undergo quadruple inversion with respect to the four circles of the tetrad taken in any order; and the several inversions being supposed performed for the twenty-four different orders in which the circles may be taken in succession; show that

(*a*) For every separate order in which the circles may be taken, the point returns after the fourth inversion to its original position in the plane.

(*b*) For the twenty-four different orders, the point and its several inverses form a group of eight points divisible into pairs of concyclic tetrads in six different ways.

Solution by the Proposer.



The property in question will manifestly be true for every quartet of coorthogonal circles in a plane, if it be true for any particular quartet into which the original may always be transformed by inversion. Hence, since by inversion from either point of intersection of any two of the three real circles of the original quartet, the system becomes transformed, as in the figure, into a pair of rectangular lines K and L, and into a pair of concentric circles M and N, the squares of the radii of M and N being equal in magnitude and opposite in sign, and their common centre O being at the intersection of K and L, and since any arbitrary point A in the plane of the transformed system becomes manifestly changed into its reflexion C with respect to O by successive inversions with respect either to K and L or to M and N, taken in each case in either order; therefore, &c., as regards the first part of the property.

The four successive inversions of A with respect to K, L, M, N being supposed taken in every possible order, the two rectangular quartets of points ABCD and EFGH are the evident result, including the original point. And, as in the transposed (and therefore in the original) con-

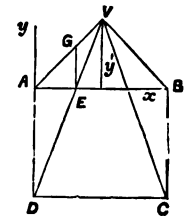
figuration, the six tetrad pairs ABCD and EFGH, AEFB and CGHD, AHGB and CFED, AEHD and BFGC, AFGD and BEHC, AEGC and BFHD are manifestly concyclic; therefore, &c., as regards the second part of the property.

Properties precisely similar, and demonstrable in precisely the same manner, hold in three dimensions for a system of five spheres, every two of which intersect at right angles, and four of which are consequently real while the fifth is imaginary; the system becoming manifestly transposed, by inversion from either point of intersection of any three of the real spheres, into a triad of rectangular planes and a pair of concentric spheres, the squares of the radii of the latter being equal in magnitude and opposite in sign, and their common centre coinciding with the intersection of the planes; and the analogous properties holding, as above, evidently for the transposed, and being consequently true for the original system.

7328. (By the Editor.)—Show that the locus of the vertex of a triangle on a given base, when the centre of the inscribed square moves on a given conic, is another conic.

Solution by D. Edwardes; J. O'Regan; *and others.*

Let AB ($=a$) be the given base, on which construct the square ABCD. Join the vertex V with D, as in the figure, cutting AB in E. Then GE, perpendicular to AB, is a side of the inscribed square. Let A be the origin, (x', y') the coordinates of V, and (x, y) those of the centre of the inscribed square. Then we have



$$\frac{GE}{y'} = \frac{AG}{AV} = \frac{DE}{DV} = \frac{a}{a+y'}, \text{ therefore } GE = \frac{ay'}{a+y'}.$$

Again, $\frac{AE}{x'} = \frac{GE}{y'}$; therefore $AE = \frac{ax'}{a+y'}$.

The locus of (x', y') is a conic, since

$$x = AE + \tfrac{1}{2}GE = \frac{a(2x'+y')}{2(a+y')}, \quad y = \tfrac{1}{2}GE = \frac{ay'}{2(a+y')}.$$

4417. (By S. Roberts, M.A.)—Two conics have a common focus, about which one is turned; show that the envelope of the common chords is a conic having a focus at the given focus.

Solution by W. J. C. Sharp, M.A.

The two conics with a common focus are the polar reciprocals of a pair of circles, the one fixed and the other revolving about a fixed point; and the required envelope is the polar reciprocal of the locus of the intersections of the common tangents.

If the equations to the circles be

$$(x-b)^2+y^2=c^2, \text{ and } (x-h)^2+(y-k)^2=r^2,$$

where $h^2 + k^2 = p^2$, the centre of similitude (ξ, η) lies on the line

$$y = \frac{k}{h-b}(x-b), \text{ and } \xi - b : \xi - h = c : r;$$

hence, by elimination, $c^2p^2 = (c-r)^2(\xi^2+\eta^2) + 2r(c-r)\xi b + r^2b^2$, a circle; and therefore the required envelope is a conic with a focus at the given focus.

Quests. 6964, 7081, 7120. *Solution by the* PROPOSER.*

(6964.)—All that I have done, or am likely to do, towards the solution of this question, is to construct the 343 24-edra, which, having only triad summits, and no faces less than 5-gons nor greater than 8-gons, reduce by the rules to the solid 5^{12}. Those rules of construction and reduction, exact converses of each other, are clearly laid down in a short tract of mine, which our learned Mathematical EDITOR has kindly undertaken to supply for the asking to any of his contributors. It fills a few pages of the *Proceedings* (1882-3) *of the Literary and Philosophical Society of Liverpool*. This process of construction of n-edra on any $(n-e)$ edron as base is definite and exhaustive, with repetitions few and all foreseen.

The figures and labels of the 343 solids are before me; but it is not worth the while, even if space permitted, to give them here. Such detail may come to be of interest, but not until a few Professors have really walked into this subject,—a walk rather long and arduous, which will not come off before somebody's day.

Under the signatures of the 24 faces, there are, among these 343, under

$8^4 5^{20}$,	1;	$8^3 6^3 5^{18}$,	2;	$8^3 7 6 5^{19}$,	2;
$8^2 7^2 6^2 5^{18}$,	12;	$8^2 7 6^4 5^{17}$,	15;	$8^2 6^6 5^{16}$,	7;
$8 7^4 6 5^{18}$,	2;	$8 7^3 6^3 5^{17}$,	26;	$8 7^2 6^5 5^{16}$,	41;
$8 7 6^7 5^{15}$,	47;	$8 6^9 5^{14}$,	9;	$7^6 5^{18}$,	3;
$7^5 6^2 5^{17}$,	4;	$7^4 6^4 5^{16}$,	33;	$7^3 6^6 5^{15}$,	55;
$7^2 6^8 5^{14}$,	70;	$7 6^{10} 5^{13}$,	13;	$6^{12} 5^{12}$,	1.

Of these 343, 59 are symmetrical, having faces of which some are either repetitions or reflected images of others. Many species of polyedral symmetry are found in the 59, but by no means all. The number of different pentagons on the 343 solids is 4896, giving so many different smoking caps for the Sultan of Borneo, constructible on the fundamental 5^{12}. The number of 24-edra built on the fundamentals $6^2 5^{12}$, $7^2 5^{14}$, $8^2 5^{16}$, &c., may turn our hundreds into thousands. Hence the benevolent reader can have no sufficient excuse, at least for the next few milleniums, for writing to that potentate on this painful subject.

Of these 343 solids, 217 are either once only made, or twice made (or makeable on another fundamental). Only one is above four times makeable by the rules.

(7081 & 7120.)—For the solution of MINERVA's riddle in Quest. 7081, we have to construct n-edra having only triad summits, and only 5-gonal and 6-gonal faces, on which every edge 55 is in at least one summit 555. I need not say that about any edge the dancers are the faces forming the edge, while the singers are the two faces which the edge meets at its two summits.

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* The Questions, being long, are given in *Contents* only.

1. After the fundamental $6^{3}5^{12}$, the first solution, for five heroes, is constructed (not regularly) by placing a point on every edge 64 of a 5-sided prism, and crowning the two wallings $6_16_16_16_16_1$ with frames 5_6. The solid formed is A_{17}, $6^{5}5^{12}$ a 5-zoned Monarchaxine Heterozone. The zones are $6\mathbf{5}0^{2}\mathbf{0}^{3}$, $\cdot\mathbf{5}0^{5}$; the two 5-zoned poles are 5-gons on the janal principal axis, and the five secondary heteroid axes carry each for poles a 2-zoned 6-gon and a 2-zoned edged 66.

2. If we treat either polar 5-gon of A_{17} as we treated the polar 5-gons of the prism, we get B_{22}, $6^{10}5^{12}$, a 5-zoned Monarchaxine Homozone, whose frames are as before, 5_6, 5_6; a solution for ten heroes.

There are six janal axes, one principal and 5-zoned with polar 5-gon and five identical zones $8\mathbf{6}0^{2}0^{4}$, and five zoneless 2-ple secondary axes each carrying polar edges 6, all ten alike. The operation can be repeated on B_{22}, giving C_{27}, on which in like manner is built D_{32}, &c. Thus for every value of m we construct the Monarchaxine Heterozones A_{17}, $C_{27}\ldots E_{17}+10m$, and the Monarchaxine Homozones B_{22}, D_{32}, $\ldots F_{22}+10m$; which have the frames 5_6, 5_6, and are all solutions of our riddle, having at least one summit 555 on every edge 55.

3. Placing a point on every edge 65 of any 6-gon of A_{17}, we have the walling $6_16_16_06_16_16_0$. On this we impose the frame 2_4, and thus construct, properly by a least frame, G_{20}, $6^{8}5^{12}$, a 3-zoned Monarchaxine Heterozone. The frames are 2_4, 2_4, 2_4, the zones are three $2\mathbf{6}0^{5}\mathbf{0}$, about the janal principal axis whose poles are 3-zoned 6-gons, and a fourth secondary zone, $6\mathbf{6}0^{3}\mathbf{0}^{3}$, met by three secondary two-zoned heteroid axes, whose zoned polar edges are on each 55 and 66. This G_{20} is a solution for eight heroes of this riddle of PALLAS, every edge 55 having a summit 555.

4. On alternate edges 64 of a six-sided prism, place six ray-points two and two, and in the opposite 6-gon do the like, making no 7-gon and leaving no 4-gon. Crowning the two wallings $6_26_06_26_06_26_0$ with frames 4_{0006}, 4_{0006}, we get H_{18}, $6^{6}5^{12}$, a Monarchaxine Homozone, having 3-zoned polar triaces on its principal janal axes, and on its three janal 2-ple secondaries six zoneless polar edges 66 all alike, which we write $(66)^{2}=6$. The zone is $4\mathbf{6}0^{4}\mathbf{0}^{2}$. Every edge 55 is in a summit 555.

5. If we place our six points so as to make two contours 210210, instead of 202020, leaving no 4-gon, we can impose two frames 4_{006}; and we thus complete I_{18}, $6^{6}5^{12}$, a zoneless Triaxine. There are three different 2-ple janal axes, whose six poles are two 2-ple 6-gons, two edges 66 and two edges 55. H_{18} and I_{18} are solutions for six heroes.

7. Remove a frame of H_{18}, and plant a 6-gon on every zero edge of the contour. The contour is still 202020, and the frame can be reimposed. This gives us a solution for nine heroes, J_{21}, $6^{9}5^{12}$, a 3-zoned Monarchaxine Heterozone, whose frames are 4_{0006}, 4_{0006}, and zones $4\mathbf{7}0^{5}\mathbf{0}^{2}$ and $6\mathbf{3}\cdot\mathbf{0}^{3}$. Repeating this on J_{21}, we get K_{24}, $6^{12}5^{12}$, a 3-zoned Monarchaxine Homozone; $4\mathbf{8}0^{6}\mathbf{0}^{2}$, $(66)^{2}=6$. Thus we can continue to augment the 6-gons by three without limit, retaining the frames 4_{0006}, 4_{0006}, on our Monarchaxine Heterozones and Homozones.

8. Removing a frame of I_{18}, we can plant two 6-gons on the two zero edges of the contour, and replacing the frame, obtain for the case of eight heroes I_{20}, a zoneless Triaxine, having still the frames 4_{006}, 4_{006}. And thus, retaining these frames, we can increase the 6-gons on these Triaxines by two without limit. All these results in (7) and (8) are solutions of PALLAS'S riddle, for squads of heroes innumerable.

9. There is also a $6^{12}5^{12}$, a zoneless 3-ple Monarchaxine, whose two principal janal poles are zoneless 3-ple triaces 666, and whose three 2-ple heteroid axes, all alike, carry each zoneless polar edges 66 and 55. The

frames are 2_4, 2_4, 2_4. The student will easily construct this solution of our riddle for 12 heroes, if it is worth his while to try.

10. Inside a 6-gon P draw faces 656565, and three more 6-gons symmetrically collateral with those just drawn and with an eighth. There are three contours 101010. Crowning them, and also P, with 1_3, 1_3, 1_3, 1_3, so as to make no 7-gon, we get M_{zz}, $6^{10}5^{12}$, $4\mathbf{6}0^40^2$, a zoned Tetrarchaxine. The four principal heteroid axes, all alike, have for poles a 3-zoned 6-gon and a triace 555; and six two-zoned 6-gons terminate the three janal secondary 2-zoned axes.

This is a solution for ten heroes, not only of MINERVA's riddle 7081, but also of APOLLO's in Quest. 7120; for every edge 55 is in one and only in one summit 555.

11. A second solution of Quest. 7120 is got by crowning every 6-gon of the zoned Tetrarchaxine 6^45^{12} by 1_3, so as to make no 7-gon. The solid formed is a zoneless Tetrarchaxine $6^{12}5^{12}$, having seven zoneless axes, four 3-ple and three 2-ple, the latter being janal axes.

A third solution is obtained by crowning with 1_3 every principal polar 6-gon on M_{22}, so as to make no 7-gon. The solid formed is $6^{18}5^{12}$ a zoned Tetrarchaxine, having seven zoned axes.

12. The solid of URANIA in Quest. 7120 is the zoned Hexarchaxine $6^{30}5^{12}$ having six 5-zoned, ten 3-zoned, and fifteen 2-zoned axes. On every face of 5^{12} place a 5-sided prism; then efface the edges of 5^{12}. The reader easily sees,

"That every four, that glorious night,
Would be heroes three, and a goddess bright."

With the help of the tract mentioned in the Solution of Quest. 6964, what is written above will be readily intelligible.

6947. (By G. F. MERRITT, M.A.)—The tangent and normal to an ellipse at the point P cut the axes in T and t, G and g, respectively; GT, gt are bisected in O and o; Oo cuts the normal Pg in Q; show that PQ is one-half of the radius of curvature at P.

Solution by R. KNOWLES, B.A.; Prof. HADAMARD; *and others.*

If P be (x_1y_1), we find

$$Ct = \frac{a^2}{x_1}; \quad CT = \frac{b^2}{y_1}; \quad Cg = e^2x; \quad Cg = -\frac{a^2e^2}{b^2}y_1; \quad CG = e^2x_1;$$

$$OC = \tfrac{1}{2}\left(\frac{b^4 - a^2e^2y_1^2}{b^2y_1}\right); \quad Co = \tfrac{1}{2}\left(\frac{a^2 + e^2x_1^2}{x_1}\right);$$

equation to $$Oo\ \frac{2xx_1}{a^2 + e^2x_1} + \frac{2b^2yy_1}{b^4 - a^2e^2y_1^2} = 1 \quad \ldots\ldots(1);$$

and from this and equation to normal we obtain

$$x = \frac{x_1(a^2 + e^2x_1^2)(b^4 + a^2e^2y_1^2)}{2a^2b^2(a^2 - e^2x_1^2)} = \frac{x_1(a^2 + e^2x_1^2)}{2a^2} \text{ and } x - x_1 = \frac{x_1(a^2 - e^2x_1^2)}{2a^2},$$

Now $$PQ^2 = (y - y_1)^2 + (x - x_1)^2 = \frac{a^4y_1^2 + b^4x_1^2}{b^4x_1^2}(x - x_1)^2$$

$$= \frac{a^2b^2(a^2 - e^2x_1^2)}{b^4x_1^2} \cdot \frac{x_1^2(a^2 - e^2x_1^2)^2}{4a^4} = \frac{(a^2 - e^2x_1^2)^3}{4a^2b^2};$$

therefore PQ = ½ radius of curvature at P.

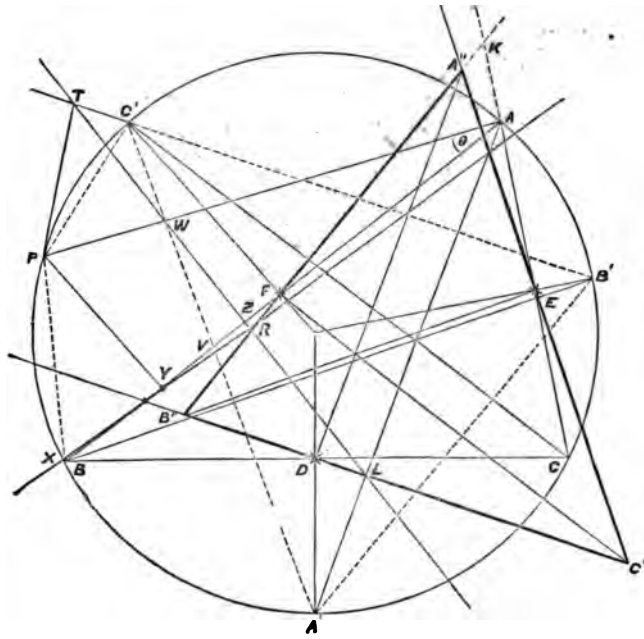
7186. (By R. Tucker, M.A.)—If A′, B′, C′, (D, E, F) are the mid-points respectively of the arcs (and sides) BC, CA, AB of a circle (and triangle) ABC; prove that (1) the pedal lines of A′, B′, C′ with reference to the triangle ABC form a triangle A″B″C″ whose sides are equal and parallel to the sides of A′B′C′, and A″D, B″E, C″F meet in the ortho-centre of A″B″C″; and (2) if P, P′ are any two diametral points, their four pedal lines with reference to the triangles ABC, A′B′C′ form a rectangle.

Solution by the Proposer.

Since $\angle A'' = EKF + A''EK = AC'F + A''EK$

$= (90° - \frac{1}{2}C) + 90° - (A + \frac{1}{2}B) = 90° - \frac{1}{2}A,$

therefore A″B″C″ is a triangle of the same species with A′B′C′, and also the triangle formed by joining points of contact of the circle inscribed in the triangle ABC.



Again $\frac{1}{2}a = EF = 2R' \sin(90° - \frac{1}{2}A) = 2R' \cos\frac{1}{2}A = R \sin A,$

therefore $R' = R \sin\frac{1}{2}A.$

Now $A''F = 2R' \cos\frac{1}{2}B = 2R \sin\frac{1}{2}A \cos\frac{1}{2}B$ .....(α).

Similarly $B''F = 2R \sin\frac{1}{2}B \cos\frac{1}{2}A,$

therefore $A''B'' = 2R \sin\frac{1}{2}(A + B) = 2R \cos\frac{1}{2}C = A'B'.$

Again $\angle ALC'' = \angle C''DC + B + \frac{1}{2}A = (90° - B - \frac{1}{2}A) + B + \frac{1}{2}A = 90°$, therefore AA′ is perpendicular to B″C″, and it is clearly perpendicular to B′C′, hence sides of A″B″C″ are parallel to the corresponding sides of A′B′C′.

It follows at once from (a) that C″F is perpendicular to A″B″; hence A″D, B″E, C″F are perpendiculars on the opposite sides, and therefore they meet in the orthocentre of A″B″C″.

The Δ A″B″C″ is easily constructed thus: let fall perpendiculars from D, E, F on the bisectors AA′, BB′, CC′; these three lines intersect in the centre of the circumscribed circle of ABC and in the orthocentre of A′B′C′. It follows that ABC, A″B″C″ have a common nine-point circle which is equal to that of A′B′C′.

(2) Let $\angle PAB = \theta$, and PX, PY, PW, PT be perpendiculars on BC, BA, A′C′, B′C′ respectively.

Now $\angle RYZ = \angle BYX = \angle BPX = 90° - (A + \theta)$,

and
$$\begin{aligned}\angle RZY &= \angle VWZ + WVZ = \angle C'WT + WVZ \\ &= [90° - (90° - \tfrac{1}{2}A + \tfrac{1}{2}C - \theta)] + [180° - (90° + \tfrac{1}{2}B)] \\ &= \tfrac{1}{2}A - \tfrac{1}{2}C + \theta + 90° - \tfrac{1}{2}B = A + \theta,\end{aligned}$$

therefore $\angle WRY$ equals 90°; that is, the pedal lines cut at right angles, and, by 1649 (*Reprint*, Vol. iii., p. 58), pedal lines, for diametral points, with respect to the same triangle cut at right angles. Hence the four pedal lines in question form a rectangle.

7322. (By Professor SYLVESTER, F.R.S.)—Prove that the number of sequences of consecutive numbers (a "sequence" of one term not excluded) having the sum n, is equal to the number of odd divisors of n.

Ex.—The odd divisors of 9 are 1, 3, 9, and 9 is represented by the three sequences 2, 3, 4; 4, 5; 9. The odd divisors of 15 are 1, 3, 5, 15, and 15 is represented by the four sequences 1, 2, 3, 4, 5; 4, 5, 6; 7, 8; 15.

[The extension of the theorem is as follows: n can be represented by *pairs* of sequences of consecutive numbers having no common term in as many ways as there are ways of satisfying the equation in positive integers $x(2y+1) + z(2t+1) = n$. Again, n can be represented by as many *triads* of sequences having no common term as there are solutions or positive integers of the equation $x(2y+1) + z(2t+1) + u(2v+1) = n$, and so on.]

Solution by W. J. C. SHARP, M.A.

If a be the first number of the sequence, and p the number of terms of which it consists, $[2a + (p-1)]\frac{1}{2}p = n$;

and, if p be odd, $2a + p - 1$ is even; or, if p be even, $2a + p - 1$ is odd. Therefore the number of solutions of the above equation is the same as the number of solutions of $x(2y+1) = n$, where, since $X = \frac{n}{2y+1}$, the number of solutions is the number of odd divisors of n.

The extensions of the question may be proved in the same way.

6993. (By G. Heppel, M.A.)—A conic passes through A, B, C. The tangents at those points intersect in O, P, Q. They intersect the opposite chords in T, U, V, which are in a straight line. Show that

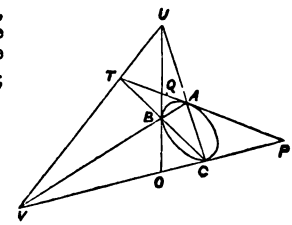
$$\pm\frac{1}{\mathrm{AUV}}\pm\frac{1}{\mathrm{BVT}}\pm\frac{1}{\mathrm{CTU}}+\frac{4}{\mathrm{OPQ}}=\frac{1}{\mathrm{ABC}},$$

the upper sign being taken where the lines AB, AC, &c. are produced both forwards or both backwards to meet the line TUV, the lower sign where one is produced forwards and the other backwards.

Solution by the Proposer.

Let ABC be the triangle of reference, let Δ be its area. Then the chords are $\alpha=0$, $\beta=0$, $\gamma=0$, the tangents are $m\beta+n\gamma=0$, $n\gamma+l\alpha=0$, $l\alpha+m\beta=0$; and the line through TUV is

$$l\alpha+m\beta+n\gamma=0.$$



$$\mathrm{POQ}=\Delta.abc\begin{vmatrix}0&m&n\\l&0&n\\l&m&0\end{vmatrix}^2+\begin{vmatrix}a&b&c\\0&m&n\\l&0&n\end{vmatrix}\begin{vmatrix}a&b&c\\l&0&n\\l&m&0\end{vmatrix}+\begin{vmatrix}a&b&c\\l&m&0\\0&m&n\end{vmatrix}$$

$$\mathrm{AUV}=\Delta\,abc\begin{vmatrix}0&1&0\\0&0&1\\l&m&n\end{vmatrix}^2\div\begin{vmatrix}a&b&c\\0&1&0\\0&0&1\end{vmatrix}\begin{vmatrix}a&b&c\\0&0&1\\l&m&n\end{vmatrix}\begin{vmatrix}a&b&c\\l&m&n\\0&1&0\end{vmatrix};$$

write u for amn, v for bnl, w for clm.

Then
$$\frac{\Delta.abc}{\mathrm{POQ}}=\frac{(u+v-w)(v+w-u)(w+u-v)}{4l^2m^2n^2},$$

$$\frac{\Delta.abc}{\mathrm{AUV}}=\frac{a(lb-ma)(lc-na)}{l^2}=\frac{u^3-u^2v-u^2w}{l^2m^2n^2}+abc;$$

therefore
$$\frac{1}{\mathrm{AUV}}+\frac{1}{\mathrm{BVT}}+\frac{1}{\mathrm{CTU}}+\frac{4}{\mathrm{OPQ}}=\frac{1}{\Delta}.$$

It remains to be determined how the signs of the areas should be chosen. If the values for AUV, BVT, CTU be added, the result is that of ABC. But from the figure it appears that it is here necessary to take AUV negative, and the others positive. Hence the rule may be obtained by consideration of different cases geometrically.

6937. (By T. Woodcock, B.A.)—A ray of light passes through a medium in which the refractive power at any point is a single-valued function of the distance from a fixed point; show that all the normals that can be drawn from the fixed point to the path of the ray are situated at equal angular intervals, and have one of two lengths alternately.

Solution by the PROPOSER; KATE GALE; *and others.*

In *any* medium let a ray take a certain path from A to B; then, if its direction be reversed at B, it will follow the same path back to A.

And if from any point two rays start in opposite directions, each at right angles to the radius vector from the fixed point, they will describe similar and equal paths in a medium such as is mentioned above.

Therefore the parts of the path of any ray on each side of an apse are similar and equal, and the proposition follows.

[The emission theory explains refraction, and will guide us aright in a question like this. If, then, we imagine a particle to describe any path through this medium with (velocity always proportional to the refractive index and) *central force* depending only on the length of the radius vector, we have simply to mention well-known properties of apses in Particle Dynamics.]

7324. (By Professor WOLSTENHOLME, M.A.) — A triangle A'B'C', whose angles are given, circumscribes a given triangle ABC; prove that there will be two such triangles of maximum area; and, if these areas be Δ_1, Δ_2, and R the radius of the circle ABC,

$$\Delta_1 \sim \Delta_2 = 8R^2 \sin A \sin B \sin C,$$

$$\Delta_1 + \Delta_2 = 4R^2 (\sin^2 A \cot A' + \sin^2 B \cot B' + \sin^2 C \cot C').$$

[If the *sign* of the areas be reckoned according to the usual rule (that A'B'C' is positive when, in going round the triangle in the order A', B', C', the area is on the left hand), the smaller of these will be negative, and with this convention $\Delta_1 + \Delta_2 = 8R^2 \sin A \sin B \sin C$,

$$\Delta_1 - \Delta_2 = 4R^2 (\sin^2 A \cot A' + \ldots + \ldots).]$$

Solution by W. S. MCCAY, M.A.

When A'B'C' is a maximum, perpendiculars to its sides at A, B, C intersect in one or other of two fixed points O or O' (see Question 7067; also TOWNSEND'S *Modern Geometry*), at which the sides of A'B'C' subtend angles $A \pm A'$, $B \pm B'$, $C \pm C'$, the upper sign being taken for the fixed point O that is inside the circumscribed circle A'B'C', and the lower for the point O' that is outside. In the first case, the vertices ABC, A'B'C' are in the same cyclical order; in the second, different. The points O, O' subtend equal or supplemental angles at the sides of ABC, and so are diametrically opposite on an equilateral hyperbola through ABC. The area of A'B'C' is

$$\tfrac{1}{2}\Sigma \frac{ab \sin (C \pm C')}{\sin A' \sin B'}, \text{ which reduces to } \pm 2\Delta \cdot ABC + \tfrac{1}{2}\Sigma a^2 \cot A',$$

which gives the result in question.

If A'B'C' and ABC are equiangular, O is at the orthocentre of ABC; O' is any point on the circumscribed circle of ABC; and $\Delta_1 = 4\Delta$ and $\Delta_2 = 0$.

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